Bessel Functions

Project for the Penn State - Göttingen Summer School on Number Theory

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Content

§ 1 The Bessel differential equation

In the first chapter, we introduce the (modified) Bessel differential equation and deduce from it the (modified) Bessel functions of first and second kind. This will be done via a power series approach. We decided to start from a differential equation, since this seems kind of naturally. We will point out later that we could have defined the Bessel function in an other, equivalent, way.

1.1 Bessel functions of first and second kind

In this section we study the the so called Bessel equation

$$x^2y'' + xy' + (x^2 - n^2)y = 0$$

for $n \in \mathbb{N}$, which can for example be obtained by the separation of the wave equation in cylindric or polar coordinates. Since this is a second order differential equation, we will have two linearly independent solutions. We will now start to construct a first solution. We take as approach for a solution y a power series. It will be convenient to write

$$y(x) = x^n \sum_{k=0}^{\infty} b_k x^k.$$

Then we have

$$xy'(x) = nx^n \sum_{k=0}^{\infty} b_k x^k + x^n \sum_{k=0}^{\infty} kb_k x^k$$

and

$$x^2y''(x) = n(n-1)x^n \sum_{k=0}^{\infty} b_k x^k + 2nx^n \sum_{k=0}^{\infty} kb_k x^k + x^n \sum_{k=0}^{\infty} k(k-1)x^k.$$

We further have

$$x^2y(x) = x^n \sum_{k=2}^{\infty} b_{k-2} x^k$$

when setting $b_{-2} := b_{-1} := 0$. Plugging in the differential equation yields

$$\begin{aligned} 0 = n(n-1)x^n \sum_{k=0}^{\infty} b_k x^k + 2nx^n + nx^n \sum_{k=0}^{\infty} b_k x^k \\ + x^n \sum_{k=0}^{\infty} kb_k x^k + x^n \sum_{k=2}^{\infty} b_{k-2} x^k - n^2 x^n \sum_{k=0}^{\infty} b_k x^k \end{aligned}$$

$$= x^n \sum_{k=0}^{\infty} (n(n-1)b_k + 2nkb_k + k(k-1)b_k + nb_k + kb_k + b_{k-2} - n^2 b_k)x^k$$
$$= x^n \sum_{k=0}^{\infty} (2nkb_k + k^2 b_k + b_{k-2})x^k.$$

Comparing coefficients gives

$$k(k+2n)b_k + b_{k-2} = 0.$$

Since $b_{-1} = 0$ we get $b_1 = -\frac{b_{-1}}{1+2n} = 0$ and recursively $b_{2k-1} = 0$ for all k as long as $2n$ is not a negative whole number. But in this case the recursion will also be fullfilled if $b_{2k-1} = 0$ for all k. So it remains to consider even k. Here we have no condition on b_0. We can choose it freely and will do so later. Now if $-n \notin \mathbb{N}$ we get

$$b_{2k} = -\frac{b_{2k-2}}{4k(n+k)}$$

and this yields inductively

$$b_{2k} = (-1)^k \frac{b_0}{4^k k!(n+k)(n+k-1)\cdots(n+1)}.$$

It is convenient to choose $b_0 = \frac{1}{2^n n!}$. So together we get

$$y(x) = \left(\frac{x}{2}\right)^n \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(n+k)!} \left(\frac{x}{2}\right)^{2k}.$$

This is convergent due to the quotient criterion, therefore we have indeed constructed a solution for the Bessel equation.

We also want to construct a solution for complex order ν. Here we need to exchane the factorial with the Gamma function. Then we get

$$J_\nu(x) = \left(\frac{x}{2}\right)^\nu \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(k+1)\Gamma(\nu+k+1)} \left(\frac{x}{2}\right)^{2k}$$

This is valid unless $-n \in \mathbb{N}$. But in this case we can just write

$$J_{-n}(x) := \left(\frac{x}{2}\right)^{-n} \sum_{k=n}^{\infty} \frac{(-1)^k}{\Gamma(k+1)\Gamma(-n+k+1)} \left(\frac{x}{2}\right)^{2k}$$
$$= \left(\frac{x}{2}\right)^{-n} \sum_{k=0}^{\infty} \frac{(-1)^{n+k}}{\Gamma(n+k+1)\Gamma(k+1)} \left(\frac{x}{2}\right)^{2n+2k}$$

$$= (-1)^n J_n(x)$$

and of course this solves the Bessel equation.

Now we want to determine a linearly independent solution. First, we describe how J_ν behaves if $x \to 0$. From the power series expansion, we immediately get

$$\lim_{x\to 0} J_\nu(x) = \begin{cases} 0, & \Re(\nu) > 0 \\ 1, & \nu = 0 \\ \pm\infty, & \Re(\nu) < 0, \nu \notin \mathbb{Z} \end{cases}.$$

This means that J_ν and $J_{-\nu}$ are two linearly independent solutions if $\nu \notin \mathbb{Z}$. If $\nu \in \mathbb{Z}$, they are linearly dependent. Since $(-1)^n = \cos n\pi$, the function $J_\nu(z)\cos\nu\pi - J_{-\nu}(z)$ is a solution of the Bessel equation, which vanishes if $n \in \mathbb{N}_0$. Therefore, we define

$$Y_\nu(x) = \frac{\cos(\nu\pi)J_\nu(x) - J_{-\nu}(x)}{\sin(\nu\pi)}.$$

for $\nu \notin \mathbb{Z}$. Then for $n \in \mathbb{Z}$ the limit

$$Y_n(x) := \lim_{\nu\to n} Y_\nu(x)$$

exists, as can be shown by L'Hospitals rule, and J_ν and Y_ν are two linearly independent solutions of the Bessel equation for all $\nu \in \mathbb{C}$. This can e.g. be shown be computing the Wronskian determinant.

1.2 Modified Bessel functions of first and second kind

The modified Bessel equation is given by

$$x^2y'' + xy' - (x^2 + \nu^2)y = 0.$$

Analog to the previous section we can compute a solution of this differential equation using the power series approach. This gives

$$I_\nu(x) = \left(\frac{x}{2}\right)^\nu \sum_{k=0}^{\infty} \frac{1}{\Gamma(k+1)\Gamma(\nu+1+k)} \left(\frac{x}{2}\right)^{2k}$$

As before, we search for a second, linearly independent solution. This will, for $\nu \notin \mathbb{Z}$, be given by

$$K_\nu(x) = \frac{\pi}{2} \frac{I_{-\nu}(x) - I_\nu(x)}{\sin(\nu\pi)}$$

and its limit for $\nu \to n \in \mathbb{Z}$ exists.

§ 2 Properties of the Bessel functions

In this chapter we will prove some properties of Bessel functions. There are of course more interesting facts, in particular, there are connections between Bessel functions and, e.g. Legendre polynomials, hypergeometric functions, the usual trigonometric functions and much more. These can be found in [Wat22] and [GR00]

2.1 Generating function

Many facts about Bessel functions can be proved by using its generating function. Here we want to determine the generating function.

Theorem 2.1 We have

$$e^{\frac{x}{2}(z-z^{-1})} = \sum_{n=-\infty}^{\infty} J_n(x) z^n \tag{2.1}$$

i.e. $e^{\frac{x}{2}(z-z^{-1})}$ is the generating function of $J_n(x)$.

Proof: We have

$$\begin{aligned} e^{\frac{x}{2}z} e^{-\frac{x}{2}\frac{1}{z}} &= \sum_{m=0}^{\infty} \frac{(\frac{x}{2})^m}{m!} z^m \sum_{k=0}^{\infty} \frac{(-1)^k (\frac{x}{2})^k}{k!} z^{-k} \\ &= \sum_{n=-\infty}^{\infty} \left(\sum_{\substack{m-k=n \\ m,k\geq 0}} \frac{(-1)^k (\frac{x}{2})^{m+k}}{m!k!} \right) z^n \\ &= \sum_{n=-\infty}^{\infty} \left(\sum_{k=0}^{\infty} \frac{(-1)^k}{(n+k)!k!} \left(\frac{x}{2}\right)^{2k} \left(\frac{x}{2}\right)^n \right) z^n \\ &= \sum_{n=-\infty}^{\infty} J_n(x) z^n. \end{aligned}$$

q.e.d.

We can use this generating function to prove some standard results.

Lemma 2.2 We have

$$\begin{aligned} \cos x &= J_0(x) + 2\sum_{n=1}^{\infty} (-1)^n J_{2n}(x), \\ \sin(x) &= 2\sum_{n=0}^{\infty} (-1)^n J_{2n+1}(x), \end{aligned}$$

$$1 = J_0(x) + 2\sum_{n=1}^{\infty} J_{2n}(x).$$

Proof: Directly from (2.1) with $z = e^{i\phi}, i\sin\phi = \frac{1}{2}(z - \frac{1}{z})$ we get

$$\cos(x\sin\phi)+i\sin(x\sin\phi) = e^{ix\sin\phi} = \sum_{n=-\infty}^{\infty} J_n(x)e^{in\phi} = \sum_{n=-\infty}^{\infty} J_n(x)(\cos(n\phi)+i\sin(n\phi))$$

so

$$\cos(x\sin\phi) = J_0(x) + 2\sum_{n=1}^{\infty} J_{2n}(x)\cos(2n\phi)$$

and

$$\sin(x\sin\phi) = 2\sum_{n=0}^{\infty} J_{2n+1}(x)\sin((2n+1)\phi).$$

This gives the results with $\phi = \frac{\pi}{2}$ and $\phi = 0$, respectively. q.e.d.

Lemma 2.3 We have

$$J_n(-x) = J_{-n}(x) = (-1)^n J_n(x)$$

for all $n \in \mathbb{Z}$.

Proof: To show the first equality, we make the change of variables $x \to -x, z \to z^{-1}$ in the generating function. Then the function does not change and we get

$$\sum_{n=-\infty}^{\infty} J_n(-x)z^{-n} = \sum_{n=-\infty}^{\infty} J_n(x)z^n = \sum_{n=-\infty}^{\infty} J_{-n}(x)z^{-n}$$

and comparing coefficients gives the result. The second equality follows analog with the change of variable $z \to -z^{-1}$. q.e.d.

Lemma 2.4 We have for any $n \in \mathbb{Z}$

$$\begin{aligned} 2J_n'(x) &= J_{n-1}(x) - J_{n+1}(x), \\ \frac{2n}{x}J_n(x) &= J_{n+1}(x) + J_{n-1}(x), \\ \frac{\mathrm{d}}{\mathrm{d}x}(x^n J_n(x)) &= x^n J_{n-1}(x). \end{aligned} \tag{2.2}$$

Proof: The first statement follows, when we differentiate equation (2.1) with respect to x:

$$\sum_{n=-\infty}^{\infty} J_n(x)z^n = \frac{1}{2}\left(z - \frac{1}{z}\right) e^{\frac{x}{2}(z-\frac{1}{z})} = \frac{1}{2}z \sum_{n=-\infty}^{\infty} J_n(x)z^n - \frac{1}{2z} \sum_{n=-\infty}^{\infty} J_n(x)z^n$$
$$= \sum_{n=-\infty}^{\infty} \frac{1}{2}(J_{n-1}(x) - J_{n+1}(x))z^n.$$

Comparing coeffivients yields the result.

The second statement follows completely analog when considering the derivative with respect to z.

Adding the two equations and multiplying with $x^n/2$ gives the third result. q.e.d.

With the last two lemmata we can describe some integrals with Bessel functions via a different Bessel function.

Lemma 2.5 For any $n \in \mathbb{Z}$ we have

$$\int x^{n+1} J_n(x)\,\mathrm{d}x = x^{n+1} J_{n+1}(x) \quad \text{and} \quad \int x^{-n+1} J_n(x)\,\mathrm{d}x = -x^{-n+1} J_{n+1}(x).$$

Proof: The first equation follows directly from (2.2). For the second one, we use (2.2) and lemma 2.3 to get

$$\int x^{-n+1} J_n(x)\,\mathrm{d}x = \int x^{-n+1}(-1)^n J_{-n}(x)\,\mathrm{d}x = (-1)^n x^{-n+1} J_{-n+1}(x) = -x^{-n+1} J_{n+1}.$$

q.e.d.

We have already seen, that $J_0^2(x) + 2\sum_{n=1}^{\infty} J_n^2(x) = 1$. With a bit more work, we can also deduce a result for $\sum J_{n+m}(x)J_n(x)$ with $m \neq 0$.

Lemma 2.6 We have for any $m \neq 0$

$$J_0^2(x) + 2\sum_{n=1}^{\infty} J_n^2(x) = 1 \quad \text{and} \quad \sum_{n=-\infty}^{\infty} J_{n+m}(x)J_n(x) = 0.$$

Proof: We multiply equation (2.1) with the equation where $z \to z^{-1}$ to get

$$1 = \sum_{k=-\infty}^{\infty} J_k(x)z^k \sum_{n=-\infty}^{\infty} J_n(x)z^{-n}$$
$$= \sum_{m=-\infty}^{\infty} \left(\sum_{n=-\infty}^{\infty} J_{n+m}(x)J_n(x) \right) z^m$$

$$= \sum_{n=-\infty}^{\infty} J_n^2(x) + \sum_{\substack{m\in\mathbb{Z}\\ m\neq 0}} \left(\sum_{n=-\infty}^{\infty} J_{n+m}(x) J_n(x) \right) z^n.$$

Comparing coefficients gives

$$1 = \sum_{n=-\infty}^{\infty} J_n^2(x) = J_0^2(x) + 2 \sum_{n=1}^{\infty} J_n^2(x)$$

and

$$0 = \sum_{n=-\infty}^{\infty} J_{n+m}(x) J_n(x)$$

for all $m \neq 0$. q.e.d.

Lemma 2.7 We have

$$\sum_{n\in\mathbb{Z}} J_n(x) = 1.$$

Proof: This follow by setting $z = 1$ in (2.1). q.e.d.

Lemma 2.8 We have

$$J_n(x+y) = \sum_{k\in\mathbb{Z}} J_k(x) J_{n-k}(y)$$

for all $n \in \mathbb{Z}$.

Proof: This follow through

$$\begin{aligned}
\sum_{n\in\mathbb{Z}} J_n(x+y) &= \exp\left(\frac{1}{2}(x+y)(t-t^{-1})\right) \\
&= \exp\left(\frac{1}{2}x(t-t^{-1})\right) \exp\left(\frac{1}{2}y(t-t^{-1})\right) \\
&= \sum_{k\in\mathbb{Z}} J_k(x) t^k \sum_{m\in\mathbb{Z}} J_m(y) t^m \\
&= \sum_{n\in\mathbb{Z}} \left(\sum_{k\in\mathbb{Z}} J_k(x) J_{n-k}(y) \right) t^n
\end{aligned}$$

and comparing coefficients. q.e.d.

Most of the results hold (in a similar form) also for the other three Bessel functions. Since the proofs are anlog, we will only state the results.

Theorem 2.9 (Results for $J_n(x), Y_n(x)$ and $K_n(x)$)

- The generating function for the $I_n(x)$ is $e^{\frac{x}{2}(z+z^{-1})}$, i.e.

$$e^{\frac{x}{2}(z+z^{-1})} = \sum_{n=-\infty}^{\infty} I_n(x) z^n.$$

- We have for all $n \in \mathbb{Z}$

$$\begin{aligned} Y_n(-x) &= -Y_{-n}(x) = (-1)^n Y_n(x), \\ I_{-n}(x) &= (-1)^n I_n(-x) = I_n(x), \\ K_{-n}(x) &= (-1)^n K_n(-x) = K_n(x). \end{aligned}$$

- For all $n \in \mathbb{Z}$ we have

$$\begin{aligned} Y_{n-1}(x) + Y_{n+1}(x) &= \frac{2n}{x} Y_n(x), & Y_{n-1}(x) - Y_{n+1}(x) &= 2Y_n'(x), \\ I_{n-1}(x) - I_{n+1}(x) &= \frac{2n}{x} I_n(x), & I_{n-1}(x) + I_{n+1}(x) &= 2I_n'(x), \\ K_{n+1}(x) - K_{n-1}(x) &= \frac{2n}{x} K_n(x), & K_{n-1}(x) + K_{n+1}(x) &= -2K_n'(x). \end{aligned}$$

  Indeed this and the next two sets of formulae, as well as those for $J_n(x)$, hold also for complex orders, as can be shown through the series representation.

- We have for all $n \in \mathbb{Z}$

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}(x^n Y_n(x)) &= x^n Y_{n-1}(x), \\ \frac{\mathrm{d}}{\mathrm{d}x}(x^n I_n(x)) &= x^n I_{n-1}(x), \\ \frac{\mathrm{d}}{\mathrm{d}x}(x^n K_n(x)) &= -x^n K_{n-1}(x). \end{aligned}$$

- For all $n \in \mathbb{Z}$ there holds

$$\begin{aligned} \int x^{n+1} Y_n(x)\,\mathrm{d}x &= x^{n+1} Y_{n+1}(x), & \int x^{-n+1} Y_n(x) &= -x^{-n+1} Y_{n+1}(x), \\ \int x^{n+1} I_n(x)\,\mathrm{d}x &= x^{n+1} I_{n+1}(x), & \int x^{-n+1} I_n(x) &= -x^{-n+1} I_{n+1}(x), \\ \int x^{n+1} K_n(x)\,\mathrm{d}x &= -x^{n+1} K_{n+1}(x), & \int x^{-n+1} K_n(x) &= x^{-n+1} K_{n+1}(x). \end{aligned}$$

- We have for all $m \in \mathbb{Z}, m \neq 0$

$$I_0^2(x) + 2\sum_{n=1}^{\infty}(-1)^n I_n^2(x) = 1 \quad \text{and} \quad \sum_{n=-\infty}^{\infty}(-1)^n I_{n+m}(x) I_n(x) = 0.$$

- We have

$$\sum_{n=-\infty}^{\infty} I_k(x) = e^x.$$

- It holds

$$I_n(x+y) = \sum_{k \in \mathbb{Z}} I_k(x) I_{n-k}(x).$$

2.2 Special values

In this section we want to examine how the Bessel functions look like when we plug in some special values.

Lemma 2.10 If $\nu \in \frac{1}{2} + \mathbb{Z}$, the Bessel functions are elementary functions.

1. We have

$$\begin{aligned} J_{\frac{1}{2}}(x) = Y_{-\frac{1}{2}}(x) &= \sqrt{\frac{2}{\pi x}} \sin(x), \\ J_{-\frac{1}{2}}(x) = -Y_{\frac{1}{2}}(x) &= \sqrt{\frac{2}{\pi x}} \cos(x), \\ I_{\frac{1}{2}}(x) &= \sqrt{2\pi x} \sinh(x), \\ I_{-\frac{1}{2}}(x) &= \sqrt{\frac{2}{\pi x}} \cosh(x), \\ K_{\frac{1}{2}}(x) = K_{-\frac{1}{2}}(x) &= \sqrt{\frac{\pi}{2x}} e^{-x}. \end{aligned}$$

2. There are polynomials $P_n(x), Q_n(x)$ with

$$\deg P_n = \deg Q_n = n, P_n(-x) = (-1)^n P_n(x), Q_n(-x) = (-1)^n Q_n(x)$$

such that for any $k \in \mathbb{N}_0$

$$\begin{aligned} J_{k+\frac{1}{2}}(x) &= \sqrt{\frac{2}{\pi x}} \left(P_k\left(\frac{1}{x}\right) \sin(x) - Q_{k-1}\left(\frac{1}{x}\right) \cos(x) \right), \\ J_{-k-\frac{1}{2}}(x) &= (-1)^k \sqrt{\frac{2}{\pi x}} \left(P_k\left(\frac{1}{x}\right) \cos(x) + Q_{k-1}\left(\frac{1}{x}\right) \sin(x) \right), \end{aligned}$$

$$
\begin{aligned}
Y_{k+\frac{1}{2}}(x) &= (-1)^{k-1} J_{-k-\frac{1}{2}}(x) \\
Y_{-k-\frac{1}{2}} &= (-1)^k J_{k+\frac{1}{2}}(x), \\
I_{k+\frac{1}{2}}(x) &= \sqrt{\frac{2}{\pi x}} \left(i^k P_k \left(\frac{i}{x} \right) \sinh(x) + i^{k-1} Q_{k-1} \left(\frac{i}{x} \right) \cosh(x) \right), \\
I_{-k-\frac{1}{2}}(x) &= \sqrt{\frac{2}{\pi x}} \left(P_k \left(\frac{1}{x} \right) \cosh(x) - Q_{k-1} \left(\frac{1}{x} \right) \sinh(x) \right), \\
K_{k+\frac{1}{2}}(x) = K_{-k-\frac{1}{2}}(x) &= \sqrt{\frac{\pi}{2x}} \left(i^k P_k \left(\frac{1}{ix} \right) \sin(x) + i^{k-1} Q_{k-1} \left(\frac{1}{ix} \right) e^{-x} \right).
\end{aligned}
$$

Proof: The first formulae follow directly from the series representations of J_ν and I_ν, the definitions of Y_ν and K_ν and the properties of the Γ-function. For example, we have

$$
\begin{aligned}
J_{\frac{1}{2}}(x) &= \sqrt{\frac{x}{2}} \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(k+1)\Gamma(k+\frac{3}{2})} \left(\frac{x}{2} \right)^{2k} \\
&= \sqrt{\frac{x}{2}} \sum_{k=0}^{\infty} \frac{(-1)^k}{k! \frac{(2k+1)!\sqrt{\pi}}{k! 2^{2k+1}}} \left(\frac{x}{2} \right)^{2k} \\
&= \sqrt{\frac{2}{\pi x}} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1} \\
&= \sqrt{\frac{2}{\pi x}} \sin x
\end{aligned}
$$

and

$$
Y_{-\frac{1}{2}}(x) = \frac{\cos(-\frac{\pi}{2}) J_{-\frac{1}{2}}(x) - J_{\frac{1}{2}}(x)}{\sin(-\frac{\pi}{2})} = J_{\frac{1}{2}}(x).
$$

The rest follows from the recurrence formulas of the last section. q.e.d.

Directly from the definitions, we also get

$$
I_n(x) = i^{-n} J_n(ix) \text{ and } K_n(x) = \frac{\pi}{2} i^{n-1} (J_n(ix) + i Y_n(ix)).
$$

2.3 Integral representations

In this section we will give an integral representation for each of the four Bessel functions that we have introduced. There are much more representations as can be shown here, see e.g. [Wat22] and [GR00].

Theorem 2.11 For all $n, x \in \mathbb{C}$ we have

$$J_\nu(x) = \frac{1}{\pi}\int_0^\pi \cos(x\sin t - \nu t)\,\mathrm{d}t - \frac{\sin(\pi\nu)}{\pi}\int_0^\infty e^{-x\sinh(t)-\nu t}\,\mathrm{d}t,$$
$$Y_\nu(x) = \frac{1}{\pi}\int_0^\pi \sin(x\sin t - \nu t)\,\mathrm{d}t - \frac{1}{\pi}\int_0^\infty e^{-x\sinh(t)}(e^{\nu t} + \cos(\pi\nu)e^{-\nu t})\,\mathrm{d}t,$$
$$I_\nu(x) = \frac{1}{\pi}\int_0^\pi e^{x\cos t}\cos(\nu t)\,\mathrm{d}t - \frac{\sin(\pi\nu)}{\pi}\int_0^\infty e^{-x\cosh(t)-\nu t}\,\mathrm{d}t,$$
$$K_\nu(x) = \int_0^\infty e^{-x\cosh(t)}\cosh(\nu t)\,\mathrm{d}t.$$

Proof: We use the representation $\frac{1}{\Gamma(z)} = \frac{1}{2\pi i}\int_{\gamma'} t^{-z}e^t\,\mathrm{d}t$ where γ' is some contour from $-\infty$, turning around 0 in positive direction and going back to $-\infty$. Then we get

$$J_\nu(x) = \frac{(x/2)^\nu}{2\pi i}\int_{\gamma'}\sum_{k=0}^\infty \frac{(-1)^k(x/2)^{2k}t^{-\nu-k-1}}{k!}e^t\,\mathrm{d}t = \frac{(x/2)^\nu}{2\pi i}\int_{\gamma'} t^{-\nu-1}e^{t-x^2/(4t)}\,\mathrm{d}t.$$

Let $t = \frac{x}{2}u$. Then we get

$$J_\nu(x) = \frac{1}{2\pi i}\int_{\tilde{\gamma}} u^{-\nu-1}e^{(x/2)(u-\frac{1}{u})}\,\mathrm{d}u$$

for some contour $\tilde{\gamma}$ of the same kind. Now let $u = e^w$. Then the contour γ, given as rectangular with vertices $\infty - i\pi, -i\pi, i\pi, \infty + i\pi$ without the right edge is the image of a suitable $\tilde{\gamma}$ (see figure 2.1). So we get

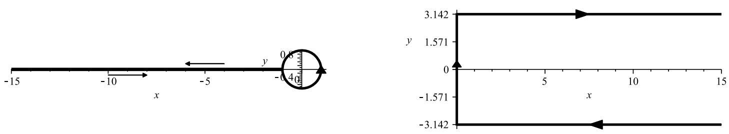

Figure 2.1: The contours for the integral.

$$J_\nu(x) = \frac{1}{2\pi i} \int\limits_\gamma e^{-\nu w} e^{x \sinh(w)} \, \mathrm{d}w = \frac{1}{2\pi i}(I_1 + I_2 - I_3)$$

with

$$\begin{aligned}
I_1 &= \int\limits_{-\pi}^{\pi} e^{-i\nu t} e^{x \sinh(it)} i \, \mathrm{d}t, \\
I_2 &= \int\limits_0^\infty e^{-\nu(i\pi + t)} e^{x \sinh(i\pi + t)} \, \mathrm{d}t, \\
I_3 &= \int\limits_0^\infty e^{-\nu(-i\pi + t)} e^{x \sinh(-i\pi + t)} \, \mathrm{d}t.
\end{aligned}$$

We have

$$\begin{aligned}
\frac{1}{2\pi i} I_1 &= \frac{1}{2\pi i} \int\limits_{-\pi}^{\pi} e^{-i\nu t} e^{x \sinh(it)} i \, \mathrm{d}t \\
&= \frac{1}{2\pi} \left(\int\limits_0^\pi e^{i(x \sin t - \nu t)} \, \mathrm{d}t + \int\limits_{-\pi}^0 e^{i(x \sin t - \nu t)} \, \mathrm{d}t \right) \\
&= \frac{1}{2\pi} \int\limits_0^\pi e^{i(x \sin t - \nu t)} + e^{-i(x \sin t - \nu t)} \, \mathrm{d}t \\
&= \frac{1}{\pi} \int\limits_0^\pi \cos(x \sin t - \nu t) \, \mathrm{d}t
\end{aligned}$$

which gives the first part. Further,

$$\begin{aligned}
\frac{1}{2\pi i}(I_2 - I_3) &= \frac{1}{2\pi i} \int\limits_0^\infty e^{-\nu(i\pi + t)} e^{x \sinh(i\pi + t)} \, \mathrm{d}t - \int\limits_0^\infty e^{-\nu(-i\pi + t)} e^{x \sinh(-i\pi + t)} \, \mathrm{d}t \\
&= \frac{1}{2\pi i} \int\limits_0^\infty e^{-x \sinh t - \nu t} (e^{-\nu i \pi} - e^{\nu i \pi}) \, \mathrm{d}t \\
&= -\frac{\sin(\pi\nu)}{\pi} \int\limits_0^\infty e^{-x \sinh t - \nu t} \, \mathrm{d}t
\end{aligned}$$

which gives the second part. This proves the first result.

For Y_ν, we note that

$$\sin(\nu\pi)Y_\nu(x) = \cos(\nu\pi)J_\nu(x) - J_{-\nu}(x) = \frac{I_1}{\pi} - \frac{\sin(\nu\pi)}{\pi}I_2$$

with

$$I_1 = \cos(\nu\pi)\int_0^\pi \cos(x\sin(t) - \nu(t))\,\mathrm{d}t - \int_0^\pi \cos(x\sin(t) + \nu t)\,\mathrm{d}t,$$

$$I_2 = \int_0^\infty e^{-x\sinh(t)}(\cos(\nu\pi)e^{-\nu t} + e^{\nu t})\,\mathrm{d}t.$$

Since

$$\cos(\nu\pi)\cos(x\sin(t) - \nu t) = \cos(x\sin(t) + \nu(\pi - t)) + \sin(\nu\pi)\sin(x\sin(t) - \nu t)$$

and

$$\int_0^\pi \cos(x\sin(t) + \nu(\pi - t))\,\mathrm{d}t = \int_0^\pi \cos(x\sin(t) + \nu t)\,\mathrm{d}t,$$

we have $I_1 = \sin(\nu\pi)\int_0^\pi \sin(x\sin(t) - \nu t)\,\mathrm{d}t$ and this proves the formula for $\nu \notin \mathbb{Z}$. Because of the continuity, we therefore get it for all ν.

The formula for I_ν is proven completely analog to that for J_ν and that for K_ν follows then from the definition, similarly to those for Y_ν. q.e.d.

With this theorem we can again follow that Y_n and K_n exist for $n \in \mathbb{Z}$.

With the generating function, these formulae can be proven even simplier, at least for $n \in \mathbb{Z}$. We will show this as an example for the functions J_n.

We use

$$\int_{-\pi}^\pi e^{int}\,\mathrm{d}t = 2\pi\delta_n^0$$

to get

$$\int_{-\pi}^\pi e^{iz\sin\theta}e^{-in\theta}\,\mathrm{d}\theta = \sum_{k\in\mathbb{Z}} J_k(x)\int_{-\pi}^\pi e^{ik\theta}e^{-in\theta}\,\mathrm{d}\theta = 2\pi J_n(x).$$

So

$$J_n(x) = \frac{1}{2\pi}\int_{-\pi}^\pi e^{i(x\sin\theta - n\theta)} = \frac{1}{\pi}\int_0^\pi \cos(x\sin\theta - n\theta)\,\mathrm{d}\theta.$$

2.4 Defining Bessel functions through the generating function

As we mentioned before, we wanted to start with the defining differential equation. We could have also defined the Bessel function $J_n(x)$ (at least for integer n) through its generating function. On this way, we also could have developed the series and integral representation:

Theorem 2.12 Let the functions $y_n(x)$ be defined by

$$e^{\frac{x}{2}(z-z^{-1})} = \sum_{n=-\infty}^{\infty} y_n(x) z^n$$

Then we have

$$y_n(x) = \left(\frac{x}{2}\right)^n \sum_{k=0}^{\infty} \frac{(-1)^k}{(n+k)!k!} \left(\frac{x}{2}\right)^{2k} \text{ and } y_n(x) = \frac{1}{\pi} \int_0^{\pi} \cos(x \sin\phi - n\phi) \,\mathrm{d}\phi.$$

Proof: First, the Cauchy integral formula gives us

$$y_n(x) = \frac{1}{2\pi i} \int_\gamma \frac{e^{\frac{x}{2}(t-t^{-1})}}{t^{n+1}} \,\mathrm{d}t$$

for any simply closed contour γ around 0.

- First we use the substitution $t = \frac{2u}{x}$ and the series representation of e^x to get

$$\begin{aligned}
y_n(x) &= \frac{1}{2\pi i} \int_\gamma \frac{e^{\frac{4u^2-x^2}{4u}}}{(\frac{2}{x})^{n+1} u^{n+1}} \frac{2}{x} \,\mathrm{d}u \\
&= \frac{1}{2\pi i} \left(\frac{x}{2}\right)^n \int_\gamma e^u e^{-\frac{x^2}{4u}} u^{-n-1} \,\mathrm{d}u \\
&= \left(\frac{x}{2}\right)^n \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^k}{m!k!} \left(\frac{x}{2}\right)^{2k} \frac{1}{2\pi i} \int_\gamma u^{m-k-n-1} \,\mathrm{d}u \\
&= \left(\frac{x}{2}\right)^n \sum_{l=0}^{\infty} \frac{(-1)^k}{(n+k)!k!} \left(\frac{x}{2}\right)^{2k}.
\end{aligned}$$

- Choosing the contour $t = e^{i\phi}$ we get

$$y_n = \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{\frac{x}{2}(e^{i\phi}-e^{-i\phi})}}{e^{ni\phi}} \,\mathrm{d}\phi$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \exp(xi\sin\phi - ni\phi)\,\mathrm{d}\phi$$
$$= \frac{1}{\pi}\int_0^{\pi} \cos(x\sin\phi - n\phi)\,\mathrm{d}\phi.$$

q.e.d.

From this integral representation we can then easily, using partial integration, see that this function solves the Bessel differential equation.

2.5 Asymptotics

Now we want to use the proven integral representations to get asymptotic formulae.

Theorem 2.13 For $x \in \mathbb{R}, x \to \infty$ we have

$$\begin{aligned} J_\nu(x) &\sim \sqrt{\frac{2}{\pi x}}\cos(x - \frac{\pi}{4} - \frac{\nu\pi}{2}), \\ Y_\nu(x) &\sim \sqrt{\frac{2}{\pi x}}\sin(x - \frac{\pi}{4} - \frac{\nu\pi}{2}), \\ I_\nu(x) &\sim \sqrt{\frac{1}{2\pi x}}e^{x}, \\ K_\nu(x) &\sim \sqrt{\frac{\pi}{2x}}e^{-x}. \end{aligned}$$

Proof:
- J_ν and Y_ν:

  The second integrals in both of the representations go to 0 exponentially, therefore we have
  $$J_\nu(x) + iY_\nu(x) = \frac{1}{\pi}\int_0^{\pi} e^{i(x\sin t - \nu t)}\,\mathrm{d}t + \mathcal{O}(x^{-A})$$
  for all A. Substitution gives
  $$J_\nu(x) + iY_\nu(x) = \frac{2e^{-i\pi\nu/2}}{\pi}(I_1(x) + I_2(x)) + \mathcal{O}(x^{-A})$$
  with
  $$I_1(x) = \int_0^{\pi/3} e^{ix\cos t}\cosh(\nu t)\,\mathrm{d}t, \qquad I_2(x) = \int_{\pi/3}^{\pi/2} e^{ix\cos t}\cosh(\nu t)\,\mathrm{d}t.$$

In I_2 we substitute $\cos(t) = u$ to get

$$I_2(x) = \int_0^{1/2} e^{ixu}\phi(u)\,\mathrm{d}u$$

with $\phi(u) = \frac{\cosh(\nu\cos^{-1}(u))}{\sqrt{1-u^2}}$. Integrating by parts give

$$I_2(x) = \frac{e^{ixu}}{ix}\phi(u)\bigg|_0^{1/2} - \frac{1}{ix}\int_0^{1/2} e^{ixu}\phi'(u)\,\mathrm{d}u = \mathcal{O}(x^{-1}),$$

since u stays away from any singularites of ϕ and ϕ' at 1.

In I_1, we set $u = \sqrt{2x}\sin(t/2)$ to get

$$I_1(x) = \frac{\sqrt{2}e^{ix}}{\sqrt{x}}\int_0^{\sqrt{x/2}} e^{iu^2}\cosh\left(2\nu\sin^{-1}\left(\frac{u}{\sqrt{2x}}\right)\right)\frac{\mathrm{d}u}{\sqrt{1-\frac{u^2}{2x}}}$$

and so

$$I_1(x) \sim \sqrt{2x}e^{ix}\int_0^{\infty} e^{it^2}\,\mathrm{d}t.$$

Using $\int_0^\infty e^{it^2}\,\mathrm{d}t = \frac{\sqrt{\pi}}{2}e^{-i\pi/4}$ we have

$$J_\nu(x) + iY_\nu(x) \sim \sqrt{\frac{2}{\pi x}}e^{i(x-pi/4-\pi\nu/2)},$$

which gives the result.

- I_ν:

  Again we have $\int_0^\infty \exp(-x\cosh t - \nu t)\,\mathrm{d}t \to 0$ so we only have to consider the other integral. We split this in two integrals, on from 0 to $\pi/2$ and the other from $\pi/2$ to π. Since $\cos(t) \le 0$ in the second interval, the second integral is bounded. For the first one we set $u = 2\sqrt{x}\sin(t/2)$ to get

$$I_\nu(x) = \frac{e^x}{\pi\sqrt{x}}\int_0^{\sqrt{2x}} e^{-\frac{u^2}{2}}\cos\left(2\nu\sin^{-1}\left(\frac{u}{2\sqrt{x}}\right)\right)\frac{\mathrm{d}u}{\sqrt{1-\frac{u^2}{4x}}} + \mathcal{O}(1).$$

  If now $x \to \infty$, the integral goes to $\int_0^\infty e^{-\frac{u^2}{2}} = \sqrt{\frac{\pi}{2}}$.

- K_ν:

  Let $u = 2\sqrt{x}\sinh(t/2)$. Then

$$K_\nu(x) = \frac{e^{-x}}{\sqrt{x}} \int\limits_0^\infty e^{-\frac{u^2}{2}} \cosh\left(2\nu \sinh^{-1}\left(\frac{u}{2\sqrt{x}}\right)\right) \frac{\mathrm{d}u}{\sqrt{1+\frac{u^2}{4x}}}.$$

  Again, for $x \to \infty$ the integral goes to $\int_0^\infty e^{-\frac{u^2}{2}} = \sqrt{\frac{\pi}{2}}$.

q.e.d.

2.6 Graphs of Bessel functions

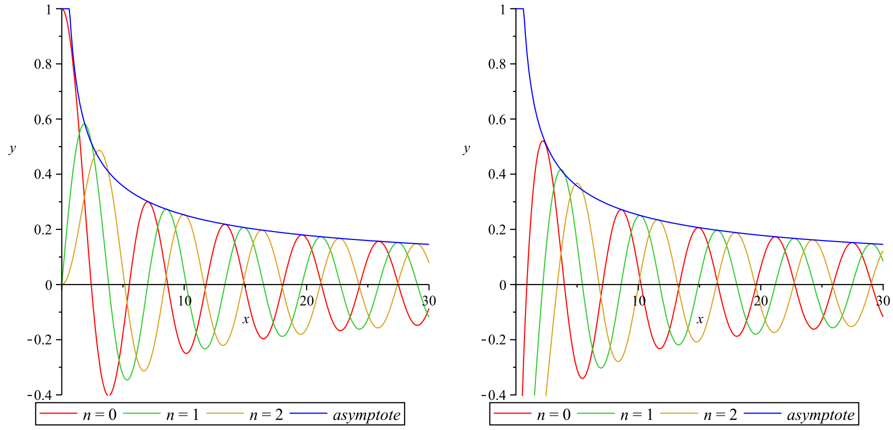

Figure 2.2: The Bessel functions $J_n(x)$ and $Y_k(x)$.

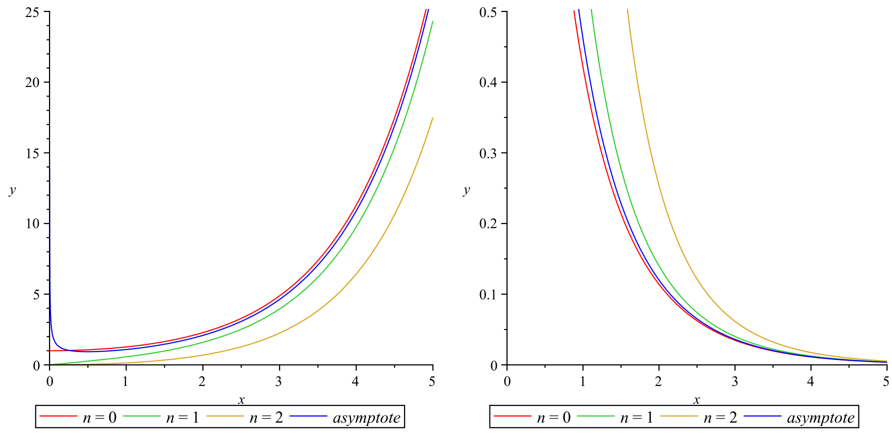

Figure 2.3: The Bessel functions $I_n(x)$ and $K_n(x)$.

2.7 Integral Transforms

In the last section, we want to compute three of the most used integral transforms. Here we will restrict to the Bessel function $J_0(x)$.

Theorem 2.14 (Laplace Transform) We have

$$\mathcal{L}[J_0](s) = \frac{1}{\sqrt{1+s^2}}.$$

Proof: The Laplace transform of the equation $x(y'' + y) + y' = 0$, which is equivalent to the Bessel equation of order 0, gives

$$\begin{aligned} 0 &= -\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{L}[y'' + y] + \mathcal{L}[y'] \\ &= -\frac{\mathrm{d}}{\mathrm{d}s}(s^2\mathcal{L}[y](s) + \mathcal{L}[y](s) - sy(0) - y'(0)) + s\mathcal{L}[y](s) - y(0) \\ &= -(1+s^2)\mathcal{L}[y]'(s) - s\mathcal{L}[y](s), \end{aligned}$$

so $\mathcal{L}[J_0](s) = \frac{c}{\sqrt{1+s^2}}$ with a constant c. We can compute c through

$$0 = \lim_{x\to\infty} \mathcal{L}[J_0'](x) = \lim_{x\to\infty} (x\mathcal{L}[J_0](x) - J_0(0)) = c - 1.$$

This means we have

$$\mathcal{L}[J_0](s) = \frac{1}{\sqrt{1+s^2}}.$$

q.e.d.

Theorem 2.15 (Fourier Transform) For $s \in \mathbb{R}$ we have

$$\mathcal{F}[J_0](s) = \begin{cases} \frac{2}{\sqrt{1-s^2}}, & |s| < 1 \\ 0, & |s| \geq 1 \end{cases}.$$

Proof: Since $J_0 \sim \sqrt{\frac{2}{\pi x}} \cos(x - \pi/4)$, the Fourier integral will diverge unless s is real. For real s we can write

$$\begin{aligned} \mathcal{F}[J_0](s) &= \lim_{\varepsilon\to 0} \int_{-\infty}^{\infty} J_0(t)e^{ist-\varepsilon|t|}\,\mathrm{d}t \\ &= \lim_{\varepsilon\to 0} \left(\int_0^{\infty} J_0(t)e^{-t(\varepsilon - is)}\,\mathrm{d}t + \int_0^{\infty} J_0(t)e^{-t(\varepsilon + is)}\,\mathrm{d}t \right) \end{aligned}$$

$$= \lim_{\varepsilon \to 0} \left(\frac{1}{\sqrt{1+(\varepsilon - is)^2}} + \frac{1}{\sqrt{1+(\varepsilon + is)^2}} \right)$$
$$= \begin{cases} \frac{2}{\sqrt{1-s^2}}, & |s| < 1 \\ 0, & |s| \geq 1 \end{cases},$$

where we have used the Laplace transform. q.e.d.

Theorem 2.16 (Mellin transform) For $0 < \Re(s) < 1$ we have

$$\mathcal{M}[J_0](s) = \frac{2^{s-1}}{\pi} \sin\left(\frac{\pi s}{2}\right) \Gamma\left(\frac{s}{2}\right)^2.$$

Proof: We use the integral representation for J_0. Since $0 < \Re(s) < 1$, we can interchange the two occuring integrals. The substitution $y = x\sin(t)$ then yields

$$\mathcal{M}[J_0](s) = \frac{1}{\pi} \int_0^\pi \int_0^\infty x^{s-1} \cos(x\sin(t))\,\mathrm{d}x\,\mathrm{d}t$$
$$= \frac{1}{\pi} \int_0^\pi \sin(t)^{-s}\,\mathrm{d}t \int_0^\infty y^{s-1} \cos(y)\,\mathrm{d}y.$$

Using

$$\int_0^\infty t^{s-1} \cos(t)\,\mathrm{d}t = \cos\left(\frac{\pi s}{2}\right)\Gamma(s) \quad \text{and} \quad \int_0^\pi \sin(t)^{-s}\,\mathrm{d}t = \sqrt{\pi}\tan\left(\frac{\pi s}{2}\right)\frac{\Gamma(\frac{s}{2})}{\Gamma(\frac{s+1}{2})}$$

and the duplication formula of the Gamma function gives

$$\mathcal{M}[J_0](s) = \frac{1}{\pi}\sqrt{\pi}\tan\left(\frac{\pi s}{2}\right)\cos\left(\frac{\pi s}{2}\right)\Gamma\left(\frac{s}{2}\right)\frac{\Gamma(s)}{\Gamma(\frac{s+1}{2})}$$
$$= \frac{2^{s-1}}{\pi}\sin\left(\frac{\pi s}{2}\right)\Gamma\left(\frac{s}{2}\right)^2.$$

q.e.d.

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