CHAPTER 1

TOPOLOGICAL VECTOR SPACES

Introduction

1.1 Many problems that analysts study are not primarily concerned with a single object such as a function, a measure, or an operator, but they deal instead with large classes of such objects. Most of the interesting classes that occur in this way turn out to be vector spaces, either with real scalars or with complex ones. Since limit processes play a role in every analytic problem (explicitly or implicitly), it should be no surprise that these vector spaces are supplied with metrics, or at least with topologies, that bear some natural relation to the objects of which the spaces are made up. The simplest and most important way of doing this is to introduce a *norm*. The resulting structure (defined below) is called a normed vector space, or a normed linear space, or simply a *normed space*.

Throughout this book, the term *vector space* will refer to a vector space over the complex field $\mathbb{C}$ or over the real field R. For the sake of completeness, detailed definitions are given in Section 1.4.

1.2 Normed spaces A vector space X is said to be a *normed space* if to every $x \in X$ there is associated a nonnegative real number $\|x\|$, called the *norm* of x, in such a way that

(*a*) $\|x + y\| \le \|x\| + \|y\|$ for all x and y in X,
(*b*) $\|\alpha x\| = |\alpha| \, \|x\|$ if $x \in X$ and α is a scalar,
(*c*) $\|x\| > 0$ if $x \ne 0$.

The word "norm" is also used to denote the *function* that maps x to $\|x\|$.

Every normed space may be regarded as a metric space, in which the distance $d(x, y)$ between x and y is $\|x - y\|$. The relevant properties of d are

(*i*) $0 \le d(x, y) < \infty$ for all x and y,
(*ii*) $d(x, y) = 0$ if and only if $x = y$,
(*iii*) $d(x, y) = d(y, x)$ for all x and y,
(*iv*) $d(x, z) \le d(x, y) + d(y, z)$ for all x, y, z.

In any metric space, the *open ball* with center at x and radius r is the set

$$B_r(x) = \{y: d(x, y) < r\}.$$

In particular, if X is a normed space, the sets

$$B_1(0) = \{x: \|x\| < 1\} \qquad \text{and} \qquad \bar{B}_1(0) = \{x: \|x\| \le 1\}$$

are the *open unit ball* and the *closed unit ball* of X, respectively.

By declaring a subset of a metric space to be open if and only if it is a (possibly empty) union of open balls, a *topology* is obtained. (See Section 1.5.) It is quite easy to verify that the vector space operations (addition and scalar multiplication) are continuous in this topology, if the metric is derived from a norm, as above.

A *Banach space* is a normed space which is *complete* in the metric defined by its norm; this means that every Cauchy sequence is required to converge.

1.3 Many of the best-known function spaces are Banach spaces. Let us mention just a few types: spaces of continuous functions on compact spaces; the familiar L^p-spaces that occur in integration theory; Hilbert spaces — the closest relatives of euclidean spaces; certain spaces of differentiable functions; spaces of continuous linear mappings from one Banach space into another; Banach algebras. All of these will occur later on in the text.

But there are also many important spaces that do not fit into this framework. Here are some examples:

(*a*) $C(\Omega)$, the space of all continuous complex functions on some open set Ω in a euclidean space R^n.

(*b*) $H(\Omega)$, the space of all holomorphic functions in some open set Ω in the complex plane.

(*c*) C_K^∞, the space of all infinitely differentiable complex functions on R^n that vanish outside some fixed compact set K with nonempty interior.

(*d*) The test function spaces used in the theory of distributions, and the distributions themselves.

These spaces carry natural topologies that cannot be induced by norms, as we shall see later. They, as well as the normed spaces, are examples of *topological vector spaces*, a concept that pervades all of functional analysis.

After this brief attempt at motivation, here are the detailed definitions, followed (in Section 1.9) by a preview of some of the results of Chapter 1.

1.4 Vector spaces The letters R and $\mathbb{C}$ will always denote the field of real numbers and the field of complex numbers, respectively. For the moment, let Φ stand for either R or $\mathbb{C}$. A *scalar* is a member of the *scalar field* Φ. A *vector space over* Φ is a set X, whose elements are called vectors, and in which two operations, *addition* and *scalar multiplication*, are defined, with the following familiar algebraic properties:

(*a*) To every pair of vectors x and y corresponds a vector $x + y$, in such a way that

$$x + y = y + x \qquad \text{and} \qquad x + (y + z) = (x + y) + z;$$

X contains a unique vector 0 (the *zero vector* or *origin* of X) such that $x + 0 = x$ for every $x \in X$; and to each $x \in X$ corresponds a unique vector $-x$ such that $x + (-x) = 0$.

(*b*) To every pair (α, x) with $\alpha \in \Phi$ and $x \in X$ corresponds a vector αx, in such a way that

$$1x = x, \qquad \alpha(\beta x) = (\alpha\beta)x,$$

and such that the two distributive laws

$$\alpha(x + y) = \alpha x + \alpha y, \qquad (\alpha + \beta)x = \alpha x + \beta x$$

hold.

The symbol 0 will of course also be used for the zero element of the scalar field.

A *real vector space* is one for which $\Phi = R$; a *complex vector space* is one for which $\Phi = \mathbb{C}$. Any statement about vector spaces in which the scalar field is not explicitly mentioned is to be understood to apply to both of these cases.

If X is a vector space, $A \subset X$, $B \subset X$, $x \in X$, and $\lambda \in \Phi$, the following notations will be used:

$$\begin{aligned} x + A &= \{x + a: a \in A\}, \\ x - A &= \{x - a: a \in A\}, \\ A + B &= \{a + b: a \in A, b \in B\}, \\ \lambda A &= \{\lambda a: a \in A\}. \end{aligned}$$

In particular (taking $\lambda = -1$), $-A$ denotes the set of all additive inverses of members of A.

A word of warning: With these conventions, it may happen that $2A \neq A + A$ (Exercise 1).

A set $Y \subset X$ is called a *subspace* of X if Y is itself a vector space (with respect to the same operations, of course). One checks easily that this happens if and only if $0 \in Y$ and

$$\alpha Y + \beta Y \subset Y$$

for all scalars α and β.

A set $C \subset X$ is said to be *convex* if

$$tC + (1 - t)C \subset C \qquad (0 \leq t \leq 1).$$

In other words, it is required that C should contain $tx + (1 - t)y$ if $x \in C$, $y \in C$, and $0 \leq t \leq 1$.

A set $B \subset X$ is said to be *balanced* if $\alpha B \subset B$ for every $\alpha \in \Phi$ with $|\alpha| \leq 1$.

A vector space X has *dimension* n ($\dim X = n$) if X has a *basis* $\{u_1, \ldots, u_n\}$. This means that every $x \in X$ has a unique representation of the form

$$x = \alpha_1 u_1 + \cdots + \alpha_n u_n \qquad (\alpha_i \in \Phi).$$

If $\dim X = n$ for some n, X is said to have *finite dimension.* If $X = \{0\}$, then $\dim X = 0$.

Example. If $X = ℂ$ (a one-dimensional vector space over the scalar field $ℂ$), the balanced sets are $ℂ$, the empty set $\varnothing$, and every circular disc (open or closed) centered at 0. If $X = R^2$ (a two-dimensional vector space over the scalar field R), there are many more balanced sets; any line segment with midpoint at (0, 0) will do. The point is that, in spite of the well-known and obvious identification of $ℂ$ with R^2, these two are entirely different as far as their vector space structure is concerned.

1.5 Topological spaces A *topological space* is a set S in which a collection τ of subsets (called *open sets*) has been specified, with the following properties: S is open, $\varnothing$ is open, the intersection of any two open sets is open, and the union of every collection of open sets is open. Such a collection τ is called a *topology on* S. When clarity seems to demand it, the topological space corresponding to the topology τ will be written (S, τ) rather than S.

Here is some of the standard vocabulary that will be used, if S and τ are as above.

A set $E \subset S$ is *closed* if and only if its complement is open. The *closure* $\bar{E}$ of E is the intersection of all closed sets that contain $\bar{E}$. The *interior* E° of E is the union of all open sets that are subsets of E. A *neighborhood* of a point $p \in S$ is any open set that contains p. (S, τ) is a *Hausdorff space*, and τ is a *Hausdorff topology*, if distinct points of S have disjoint neighborhoods. A set $K \subset S$ is *compact* if every open cover of K has a finite subcover. A collection $\tau' \subset \tau$ is a *base* for τ if every member of τ (that is, every open set) is a union of members of τ'. A collection γ of neighborhoods of a point $p \in S$ is a *local base at* p if every neighborhood of p contains a member of γ.

If $E \subset S$ and if σ is the collection of all intersections $E \cap V$, with $V \in \tau$, then σ is a topology on E, as is easily verified; we call this the topology that E *inherits* from S.

If a topology τ is induced by a metric d (see Section 1.2) we say that d and τ are *compatible* with each other.

A sequence $\{x_n\}$ in a Hausdorff space X *converges* to a point $x \in X$ (or $\lim_{n\to\infty} x_n = x$) if every neighborhood of x contains all but finitely many of the points x_n.

1.6 Topological vector spaces

Suppose τ is a topology on a vector space X such that

(*a*) *every point of X is a closed set, and*

(*b*) *the vector space operations are continuous with respect to τ.*

Under these conditions, τ is said to be a *vector topology* on X, and X is a *topological vector space.*

Here is a more precise way of stating (*a*): For every $x \in X$, the set $\{x\}$ which has x as its only member is a closed set.

In many texts, (*a*) is omitted from the definition of a topological vector space. Since (*a*) is satisfied in almost every application, and since most theorems of interest require (*a*) in their hypotheses, it seems best to include it in the axioms. [Theorem 1.12 will show that (*a*) and (*b*) together imply that τ is a Hausdorff topology.]

To say that addition is *continuous* means, by definition, that the mapping

$$(x, y) \to x + y$$

of the cartesian product $X \times X$ into X is continuous: If $x_i \in X$ for $i = 1, 2$, and if V is a neighborhood of $x_1 + x_2$, there should exist neighborhoods V_i of x_i such that

$$V_1 + V_2 \subset V.$$

Similarly, the assumption that scalar multiplication is continuous means that the mapping

$$(\alpha, x) \to \alpha x$$

of $\Phi \times X$ into X is continuous: If $x \in X$, α is a scalar, and V is a neighborhood of αx, then for some $r > 0$ and some neighborhood W of x we have $\beta W \subset V$ whenever $|\beta - \alpha| < r$.

A subset E of a topological vector space is said to be *bounded* if to every neighborhood V of 0 in X corresponds a number $s > 0$ such that $E \subset tV$ for every $t > s$.

1.7 Invariance Let X be a topological vector space. Associate to each $a \in X$ and to each scalar $\lambda \neq 0$ the *translation operator* T_a and the *multiplication operator* M_λ, by the formulas

$$T_a(x) = a + x, \qquad M_\lambda(x) = \lambda x \qquad (x \in X).$$

The following simple proposition is very important:

Proposition. *T_a and M_λ are homeomorphisms of X onto X.*

PROOF. The vector space axioms alone imply that T_a and M_λ are one-to-one, that they map X onto X, and that their inverses are T_{-a} and $M_{1/\lambda}$, respectively. The assumed continuity of the vector space operations implies that these four mappings are continuous. Hence each of them is a homeomorphism (a continuous mapping whose inverse is also continuous). ////

One consequence of this proposition is that every vector topology τ is *translation-invariant* (or simply *invariant*, for brevity): A set $E \subset X$ is open if and only if each of its translates $a + E$ is open. Thus τ is completely determined by any local base.

In the vector space context, the term *local base* will always mean a local base at 0. A local base of a topological vector space X is thus a collection $\mathscr{B}$ of neighborhoods of 0 such that every neighborhood of 0 contains a member of $\mathscr{B}$. The open sets of X are then precisely those that are unions of translates of members of $\mathscr{B}$.

A metric d on a vector space X will be called *invariant* if

$$d(x + z, y + z) = d(x, y)$$

for all x, y, z in X.

1.8 Types of topological vector spaces In the following definitions, X always denotes a topological vector space, with topology τ.

(*a*) X is *locally convex* if there is a local base $\mathscr{B}$ whose members are convex.
(*b*) X is *locally bounded* if 0 has a bounded neighborhood.
(*c*) X is *locally compact* if 0 has a neighborhood whose closure is compact.
(*d*) X is *metrizable* if τ is compatible with some metric d.
(*e*) X is an *F-space* if its topology τ is induced by a complete invariant metric d. (Compare Section 1.25.)
(*f*) X is a *Fréchet space* if X is a locally convex F-space.
(*g*) X is *normable* if a norm exists on X such that the metric induced by the norm is compatible with τ.
(*h*) *Normed spaces* and *Banach spaces* have already been defined (Section 1.2).
(*i*) X has the *Heine-Borel property* if every closed and bounded subset of X is compact.

The terminology of (*e*) and (*f*) is not universally agreed upon: In some texts, local convexity is omitted from the definition of a Fréchet space, whereas others use F-space to describe what we have called Fréchet space.

1.9 Here is a list of some relations between these properties of a topological vector space X.

(*a*) If X is locally bounded, then X has a countable local base [part (*c*) of Theorem 1.15].
(*b*) X is metrizable if and only if X has a countable local base (Theorem 1.24).
(*c*) X is normable if and only if X is locally convex and locally bounded (Theorem 1.39).
(*d*) X has finite dimension if and only if X is locally compact (Theorems 1.21, 1.22).
(*e*) If a locally bounded space X has the Heine-Borel property, then X has finite dimension (Theorem 1.23).

The spaces $H(\Omega)$ and C_K^∞ mentioned in Section 1.3 are infinite-dimensional Fréchet spaces with the Heine-Borel property (Sections 1.45, 1.46). They are therefore not locally bounded, hence not normable; they also show that the converse of (*a*) is false.

On the other hand, there exist locally bounded F-spaces that are not locally convex (Section 1.47).

Separation Properties

1.10 Theorem *Suppose K and C are subsets of a topological vector space X, K is compact, C is closed, and $K \cap C = \varnothing$. Then 0 has a neighborhood V such that*

$$(K + V) \cap (C + V) = \varnothing.$$

Note that $K + V$ is a union of translates $x + V$ of V $(x \in K)$. Thus $K + V$ is an open set that contains K. The theorem thus implies the existence of disjoint open sets that contain K and C, respectively.

PROOF. We begin with the following proposition, which will be useful in other contexts as well:

If W is a neighborhood of 0 in X, then there is a neighborhood U of 0 which is symmetric (in the sense that $U = -U$) and which satisfies $U + U \subset W$.

To see this, note that $0 + 0 = 0$, that addition is continuous, and that 0 therefore has neighborhoods V_1, V_2 such that $V_1 + V_2 \subset W$. If

$$U = V_1 \cap V_2 \cap (-V_1) \cap (-V_2),$$

then U has the required properties.

The proposition can now be applied to U in place of W and yields a new symmetric neighborhood U of 0 such that

$$U + U + U + U \subset W.$$

It is clear how this can be continued.

If $K = \varnothing$, then $K + V = \varnothing$, and the conclusion of the theorem is obvious. We therefore assume that $K \neq \varnothing$, and consider a point $x \in K$. Since C is closed, since x is not in C, and since the topology of X is invariant under translations, the preceding proposition shows that 0 has a symmetric neighborhood V_x such that $x + V_x + V_x + V_x$ does not intersect C; the symmetry of V_x shows then that

$$(x + V_x + V_x) \cap (C + V_x) = \varnothing. \tag{1}$$

Since K is compact, there are finitely many points $x_1, \ldots, x_n$ in K such that

$$K \subset (x_1 + V_{x_1}) \cup \cdots \cup (x_n + V_{x_n}).$$

Put $V = V_{x_1} \cap \cdots \cap V_{x_n}$. Then

$$K + V \subset \bigcup_{i=1}^{n} (x_i + V_{x_i} + V) \subset \bigcup_{i=1}^{n} (x_i + V_{x_i} + V_{x_i}),$$

and no term in this last union intersects $C + V$, by (1). This completes the proof. ////

Since $C + V$ is open, it is even true that the *closure* of $K + V$ does not intersect $C + V$; in particular, the closure of $K + V$ does not intersect C. The following special case of this, obtained by taking $K = \{0\}$, is of considerable interest.

1.11 Theorem *If $\mathscr{B}$ is a local base for a topological vector space X, then every member of $\mathscr{B}$ contains the closure of some member of $\mathscr{B}$.*

So far we have not used the assumption that every point of X is a closed set. We now use it and apply Theorem 1.10 to a pair of distinct points in place of K and C. The conclusion is that these points have disjoint neighborhoods. In other words, the Hausdorff separation axiom holds:

1.12 Theorem *Every topological vector space is a Hausdorff space.*

We now derive some simple properties of closures and interiors in a topological vector space. See Section 1.5 for the notations $\bar{E}$ and E°. Observe that a point p belongs to $\bar{E}$ if and only if every neighborhood of p intersects E.

1.13 Theorem *Let X be a topological vector space.*

(*a*) *If $A \subset X$ then $\bar{A} = \bigcap (A + V)$, where V runs through all neighborhoods of 0.*
(*b*) *If $A \subset X$ and $B \subset X$, then $\bar{A} + \bar{B} \subset \overline{A + B}$.*
(*c*) *If Y is a subspace of X, so is $\bar{Y}$.*
(*d*) *If C is a convex subset of X, so are $\bar{C}$ and C°.*
(*e*) *If B is a balanced subset of X, so is $\bar{B}$; if also $0 \in B^\circ$ then B° is balanced.*
(*f*) *If E is a bounded subset of X, so is $\bar{E}$.*

PROOF. (*a*) $x \in \bar{A}$ if and only if $(x + V) \cap A \neq \emptyset$ for every neighborhood V of 0, and this happens if and only if $x \in A - V$ for every such V. Since $-V$ is a neighborhood of 0 if and only if V is one, the proof is complete.

(*b*) Take $a \in \bar{A}$, $b \in \bar{B}$; let W be a neighborhood of $a + b$. There are neighborhoods W_1 and W_2 of a and b such that $W_1 + W_2 \subset W$. There exist $x \in A \cap W_1$ and $y \in B \cap W_2$, since $a \in \bar{A}$ and $b \in \bar{B}$. Then $x + y$ lies in $(A + B) \cap W$, so that this intersection is not empty. Consequently, $a + b \in \overline{A + B}$.

(*c*) Suppose α and β are scalars. By the proposition in Section 1.7, $\alpha\bar{Y} = \overline{\alpha Y}$ if $\alpha \neq 0$; if $\alpha = 0$, these two sets are obviously equal. Hence it follows from (*b*) that

$$\alpha\bar{Y} + \beta\bar{Y} = \overline{\alpha Y} + \overline{\beta Y} \subset \overline{\alpha Y + \beta Y} \subset \bar{Y};$$

the assumption that Y is a subspace was used in the last inclusion.

The proofs that convex sets have convex closures and that balanced sets have balanced closures are so similar to this proof of (*c*) that we shall omit them from (*d*) and (*e*).

(*d*) Since $C^\circ \subset C$ and C is convex, we have

$$tC^\circ + (1 - t)C^\circ \subset C$$

if $0 < t < 1$. The two sets on the left are open; hence so is their sum. Since every open subset of C is a subset of C°, it follows that C° is convex.

(*e*) If $0 < |\alpha| \leq 1$, then $\alpha B^\circ = (\alpha B)^\circ$, since $x \to \alpha x$ is a homeomorphism. Hence $\alpha B^\circ \subset \alpha B \subset B$, since B is balanced. But αB° is open. So $\alpha B^\circ \subset B^\circ$. If B° contains the origin, then $\alpha B^\circ \subset B^\circ$ even for $\alpha = 0$.

(*f*) Let V be a neighborhood of 0. By Theorem 1.11, $\bar{W} \subset V$ for some neighborhood W of 0. Since E is bounded, $E \subset tW$ for all sufficiently large t. For these t, we have $\bar{E} \subset t\bar{W} \subset tV$. ///

1.14 Theorem *In a topological vector space X,*

(*a*) *every neighborhood of* 0 *contains a balanced neighborhood of* 0, *and*

(*b*) *every convex neighborhood of* 0 *contains a balanced convex neighborhood of* 0.

PROOF. (*a*) Suppose U is a neighborhood of 0 in X. Since scalar multiplication is continuous, there is a $\delta > 0$ and there is a neighborhood V of 0 in X such that $\alpha V \subset U$ whenever $|\alpha| < \delta$. Let W be the union of all these sets αV. Then W is a neighborhood of 0, W is balanced, and $W \subset U$.

(*b*) Suppose U is a convex neighborhood of 0 in X. Let $A = \bigcap \alpha U$, where α ranges over the scalars of absolute value 1. Choose W as in part (*a*). Since W is balanced, $\alpha^{-1}W = W$ when $|\alpha| = 1$; hence $W \subset \alpha U$. Thus $W \subset A$, which implies that the interior A° of A is a neighborhood of 0. Clearly $A^\circ \subset U$. Being an intersection of convex sets, A is convex; hence so is A°. To prove that A° is a neighborhood with the desired properties, we have to show that A° is balanced; for this it suffices to prove that A is balanced. Choose r and β so that $0 \le r \le 1$, $|\beta| = 1$. Then

$$r\beta A = \bigcap_{|\alpha|=1} r\beta\alpha U = \bigcap_{|\alpha|=1} r\alpha U.$$

Since αU is a convex set that contains 0, we have $r\alpha U \subset \alpha U$. Thus $r\beta A \subset A$, which completes the proof. ////

Theorem 1.14 can be restated in terms of local bases. Let us say that a local base $\mathscr{B}$ is *balanced* if its members are balanced sets, and let us call $\mathscr{B}$ *convex* if its members are convex sets.

Corollary

(*a*) *Every topological vector space has a balanced local base.*
(*b*) *Every locally convex space has a balanced convex local base.*

Recall also that Theorem 1.11 holds for each of these local bases.

1.15 Theorem *Suppose V is a neighborhood of* 0 *in a topological vector space X.*

(*a*) *If $0 < r_1 < r_2 < \cdots$ and $r_n \to \infty$ as $n \to \infty$, then*

$$X = \bigcup_{n=1}^{\infty} r_n V.$$

(*b*) *Every compact subset K of X is bounded.*
(*c*) *If $\delta_1 > \delta_2 > \cdots$ and $\delta_n \to 0$ as $n \to \infty$, and if V is bounded, then the collection*

$$\{\delta_n V : n = 1, 2, 3, \ldots\}$$

is a local base for X.

PROOF. (*a*) Fix $x \in X$. Since $\alpha \to \alpha x$ is a continuous mapping of the scalar field into X, the set of all α with $\alpha x \in V$ is open, contains 0, hence contains $1/r_n$ for all large n. Thus $(1/r_n)x \in V$, or $x \in r_n V$, for large n.

(*b*) Let W be a balanced neighborhood of 0 such that $W \subset V$. By (*a*),

$$K \subset \bigcup_{n=1}^{\infty} nW.$$

Since K is compact, there are integers $n_1 < \cdots < n_s$ such that

$$K \subset n_1 W \cup \cdots \cup n_s W = n_s W.$$

The equality holds because W is balanced. If $t > n_s$, it follows that $K \subset tW \subset tV$.

(*c*) Let U be a neighborhood of 0 in X. If V is bounded, there exists $s > 0$ such that $V \subset tU$ for all $t > s$. If n is so large that $s\delta_n < 1$, it follows that $V \subset (1/\delta_n)U$. Hence U actually contains all but finitely many of the sets $\delta_n V$. ////

Linear Mappings

1.16 Definitions When X and Y are sets, the symbol

$$f: X \to Y$$

will mean that f is a mapping of X into Y. If $A \subset X$ and $B \subset Y$, the *image* $f(A)$ of A and the *inverse image* or *preimage* $f^{-1}(B)$ of B are defined by

$$f(A) = \{f(x): x \in A\}, \qquad f^{-1}(B) = \{x: f(x) \in B\}.$$

Suppose now that X and Y are vector spaces *over the same scalar field*. A mapping $\Lambda: X \to Y$ is said to be *linear* if

$$\Lambda(\alpha x + \beta y) = \alpha\Lambda x + \beta\Lambda y$$

for all x and y in X and all scalars α and β. Note that one often writes Λx, rather than $\Lambda(x)$, when Λ is linear.

Linear mappings of X into its scalar field are called *linear functionals*.

For example, the multiplication operators M_α of Section 1.7 are linear, but the translation operators T_a are not, except when $a = 0$.

Here are some properties of linear mappings $\Lambda: X \to Y$ whose proofs are so easy that we omit them; it is assumed that $A \subset X$ and $B \subset Y$:

(*a*) $\Lambda 0 = 0$.

(*b*) If A is a subspace (or a convex set, or a balanced set) the same is true of $\Lambda(A)$.

(*c*) If B is a subspace (or a convex set, or a balanced set) the same is true of $\Lambda^{-1}(B)$.

(*d*) In particular, the set

$$\Lambda^{-1}(\{0\}) = \{x \in X: \Lambda x = 0\} = \mathscr{N}(\Lambda)$$

is a subspace of X, called the *null space* of Λ.

We now turn to continuity properties of linear mappings.

1.17 Theorem *Let X and Y be topological vector spaces. If $\Lambda: X \to Y$ is linear and continuous at 0, then Λ is continuous. In fact, Λ is uniformly continuous, in the following sense: To each neighborhood W of 0 in Y corresponds a neighborhood V of 0 in X such that*

$$y - x \in V \textit{ implies } \Lambda y - \Lambda x \in W.$$

PROOF. Once W is chosen, the continuity of Λ at 0 shows that $\Lambda V \subset W$ for some neighborhood V of 0. If now $y - x \in V$, the linearity of Λ shows that $\Lambda y - \Lambda x = \Lambda(y - x) \in W$. Thus Λ maps the neighborhood $x + V$ of x into the preassigned neighborhood $\Lambda x + W$ of Λx, which says that Λ is continuous at x. ////

1.18 Theorem *Let Λ be a linear functional on a topological vector space X. Assume $\Lambda x \neq 0$ for some $x \in X$. Then each of the following four properties implies the other three:*

(*a*) *Λ is continuous.*
(*b*) *The null space $\mathscr{N}(\Lambda)$ is closed.*
(*c*) *$\mathscr{N}(\Lambda)$ is not dense in X.*
(*d*) *Λ is bounded in some neighborhood V of 0.*

PROOF. Since $\mathscr{N}(\Lambda) = \Lambda^{-1}(\{0\})$ and $\{0\}$ is a closed subset of the scalar field Φ, (*a*) implies (*b*). By hypothesis, $\mathscr{N}(\Lambda) \neq X$. Hence (*b*) implies (*c*).

Assume (*c*) holds; i.e., assume that the complement of $\mathscr{N}(\Lambda)$ has nonempty interior. By Theorem 1.14,

$$(x + V) \cap \mathscr{N}(\Lambda) = \varnothing \tag{1}$$

for some $x \in X$ and some balanced neighborhood V of 0. Then ΛV is a balanced subset of the field Φ. Thus either ΛV is bounded, in which case (*d*) holds, or $\Lambda V = \Phi$. In the latter case, there exists $y \in V$ such that $\Lambda y = -\Lambda x$, and so $x + y \in \mathscr{N}(\Lambda)$, in contradiction to (1). Thus (*c*) implies (*d*).

Finally, if (*d*) holds then $|\Lambda x| < M$ for all x in V and for some $M < \infty$. If $r > 0$ and if $W = (r/M)V$, then $|\Lambda x| < r$ for every x in W. Hence Λ is continuous at the origin. By Theorem 1.17, this implies (*a*). ////

Finite-Dimensional Spaces

1.19 Among the simplest Banach spaces are R^n and $\mathcal{C}^n$, the standard n-dimensional vector spaces over R and $\mathcal{C}$, respectively, normed by mean of the usual euclidean metric: If, for example,

$$z = (z_1, \ldots, z_n) \qquad (z_i \in \mathcal{C})$$

is a vector in $\mathcal{C}^n$, then

$$\|z\| = (|z_1|^2 + \cdots + |z_n|^2)^{1/2}.$$

Other norms can be defined on $\mathcal{C}^n$. For example,

$$\|z\| = |z_1| + \cdots + |z_n| \qquad \text{or} \qquad \|z\| = \max\,(|z_i| : 1 \le i \le n).$$

These norms correspond, of course, to different metrics on $\mathcal{C}^n$ (when $n > 1$) but one can see very easily that they all induce the same topology on $\mathcal{C}^n$. Actually, more is true.

If X is a topological vector space over $\mathcal{C}$, and $\dim X = n$, then every basis of X induces an isomorphism of X onto $\mathcal{C}^n$. Theorem 1.21 will prove that this *isomorphism must be a homeomorphism.* In other words, this says that *the topology of $\mathcal{C}^n$ is the only vector topology that an n-dimensional complex topological vector space can have.*

We shall also see that finite-dimensional subspaces are always closed and that no infinite-dimensional topological vector space is locally compact.

Everything in the preceding discussion remains true with real scalars in place of complex ones.

1.20 Lemma *If X is a complex topological vector space and $f: \mathcal{C}^n \to X$ is linear, then f is continuous.*

PROOF. Let $\{e_1, \ldots, e_n\}$ be the standard basis of $\mathcal{C}^n$: The kth coordinate of e_k is 1, the others are 0. Put $u_k = f(e_k)$, for $k = 1, \ldots, n$. Then $f(z) = z_1 u_1 + \cdots + z_n u_n$ for every $z = (z_1, \ldots, z_n)$ in $\mathcal{C}^n$. Every z_k is a continuous function of z. The continuity of f is therefore an immediate consequence of the fact that addition and scalar multiplication are continuous in X. ////

1.21 Theorem *If n is a positive integer and Y is an n-dimensional subspace of a complex topological vector space X, then*

(*a*) *every isomorphism of $\mathcal{C}^n$ onto Y is a homeomorphism, and*

(*b*) *Y is closed.*

PROOF. Let S be the sphere which bounds the open unit ball B of $\mathcal{C}^n$. Thus $z \in S$ if and only if $\Sigma|z_i|^2 = 1$, and $z \in B$ if and only if $\Sigma|z_i|^2 < 1$.

Suppose $f: \mathcal{C}^n \to Y$ is an isomorphism. This means that f is linear, one-to-one, and $f(\mathcal{C}^n) = Y$. Put $K = f(S)$. Since f is continuous (Lemma 1.20), K is compact. Since $f(0) = 0$ and f is one-to-one, $0 \notin K$, and therefore there is a balanced neighborhood V of 0 in X which does not intersect K. The set

$$E = f^{-1}(V) = f^{-1}(V \cap Y)$$

is therefore disjoint from S. Since f is linear, E is balanced, and hence connected. Thus $E \subset B$, because $0 \in E$, and this implies that the linear map f^{-1} takes $V \cap Y$ into B. Since f^{-1} is an n-tuple of linear functionals on Y, the implication $(d) \to (a)$ in Theorem 1.18 shows that f^{-1} is continuous. Thus f is a homeomorphism.

To prove (b), choose $p \in \bar{Y}$, and let f and V be as above. For some $t > 0$, $p \in tV$, so that p lies in the closure of

$$Y \cap (tV) \subset f(tB) \subset f(t\bar{B}).$$

Being compact, $f(t\bar{B})$ is closed in X. Hence $p \in f(t\bar{B}) \subset Y$, and this proves that $\bar{Y} = Y$. ////

1.22 Theorem *Every locally compact topological vector space X has finite dimension.*

PROOF. The origin of X has a neighborhood V whose closure is compact. By Theorem 1.15, V is bounded, and the sets $2^{-n}V$ ($n = 1, 2, 3, \ldots$) form a local base for X.

The compactness of $\bar{V}$ shows that there exist $x_1, \ldots, x_m$ in X such that

$$\bar{V} \subset (x_1 + \tfrac{1}{2}V) \cup \cdots \cup (x_m + \tfrac{1}{2}V).$$

Let Y be the vector space spanned by $x_1, \ldots, x_m$. Then $\dim Y \le m$. By Theorem 1.21, Y is a *closed* subspace of X.

Since $V \subset Y + \frac{1}{2}V$ and since $\lambda Y = Y$ for every scalar $\lambda \neq 0$, it follows that

$$\tfrac{1}{2}V \subset Y + \tfrac{1}{4}V$$

so that

$$V \subset Y + \tfrac{1}{2}V \subset Y + Y + \tfrac{1}{4}V = Y + \tfrac{1}{4}V.$$

If we continue in this way, we see that

$$V \subset \bigcap_{n=1}^{\infty} (Y + 2^{-n}V).$$

Since $\{2^{-n}V\}$ is a local base, it now follows from (*a*) of Theorem 1.13 that $V \subset \bar{Y}$. But $\bar{Y} = Y$. Thus $V \subset Y$, which implies that $kV \subset Y$ for $k = 1, 2, 3, \ldots$. Hence $Y = X$, by (*a*) of Theorem 1.15, and consequently dim $X \leq m$. ////

1.23 Theorem *If X is a locally bounded topological vector space with the Heine-Borel property, then X has finite dimension.*

PROOF. By assumption, the origin of X has a bounded neighborhood V. Statement (*f*) of Theorem 1.13 shows that $\bar{V}$ is also bounded. Thus $\bar{V}$ is compact, by the Heine-Borel property. This says that X is locally compact, hence finite-dimensional, by Theorem 1.22.

Metrization

We recall that a topology τ on a set X is said to be *metrizable* if there is a metric d on X which is compatible with τ. In that case, the balls with radius $1/n$ centered at x form a local base at x. This gives a necessary condition for metrizability which, for topological vector spaces, turns out to be also sufficient.

1.24 Theorem *If X is a topological vector space with a countable local base, then there is a metric d on X such that*

(*a*) *d is compatible with the topology of X,*

(*b*) *the open balls centered at 0 are balanced, and*

(*c*) *d is invariant:* $d(x + z, y + z) = d(x, y)$ *for* $x, y, z \in X$.

If, in addition, X is locally convex, then d can be chosen so as to satisfy (*a*), (*b*), (*c*), *and also*

(*d*) *all open balls are convex.*

PROOF. By Theorem 1.14, X has a balanced local base $\{V_n\}$ such that

(1) $$V_{n+1} + V_{n+1} + V_{n+1} + V_{n+1} \subset V_n \qquad (n + 1, 2, 3, \ldots);$$

when X is locally convex, this local base can be chosen so that each V_n is also convex.

Let D be the set of all rational numbers r of the form

(2) $$r = \sum_{n=1}^{\infty} c_n(r)2^{-n},$$

where each of the "digits" $c_i(r)$ is 0 or 1 and only finitely many are 1. Thus each $r \in D$ satisfies the inequalities $0 \le r < 1$.

Put $A(r) = X$ if $r \ge 1$; for any $r \in D$, define

(3) $$A(r) = c_1(r)V_1 + c_2(r)V_2 + c_3(r)V_3 + \cdots.$$

Note that each of these sums is actually finite. Define

(4) $$f(x) = \inf \{r: x \in A(r)\} \qquad (x \in X)$$

and

(5) $$d(x, y) = f(x - y) \qquad (x \in X, y \in X).$$

The proof that this d has the desired properties depends on the inclusions

(6) $$A(r) + A(s) \subset A(r + s) \qquad (r \in D, s \in D).$$

Before proving (6), let us see how the theorem follows from it. Since every $A(s)$ contains 0, (6) imples

(7) $$A(r) \subset A(r) + A(t - r) \subset A(t) \qquad \text{if} \qquad r < t.$$

Thus $\{A(r)\}$ is totally ordered by set inclusion. We claim that

(8) $$f(x + y) \le f(x) + f(y) \qquad (x \in X, y \in X).$$

In the proof of (8) we may, of course, assume that the right side is <1. Fix $\varepsilon > 0$. There exist r and s in D such that

$$f(x) < r, \qquad f(y) < s, \qquad r + s < f(x) + f(y) + \varepsilon.$$

Thus $x \in A(r)$, $y \in A(s)$, and (6) implies $x + y \in A(r + s)$. Now (8) follows, because

$$f(x + y) \le r + s < f(x) + f(y) + \varepsilon,$$

and ε was arbitrary.

Since each $A(r)$ is balanced, $f(x) = f(-x)$. It is obvious that $f(0) = 0$. If $x \ne 0$, then $x \notin V_n = A(2^{-n})$ for some n, and so $f(x) \ge 2^{-n} > 0$.

These properties of f show that (5) defines a translation-invariant metric d on X. The open balls centered at 0 are the open sets

(9) $$B_\delta(0) = \{x: f(x) < \delta\} = \bigcup_{r<\delta} A(r).$$

If $\delta < 2^{-n}$, then $B_\delta(0) \subset V_n$. Hence $\{B_\delta(0)\}$ is a local base for the topology of X. This proves (a). Since each $A(r)$ is balanced, so is each $B_\delta(0)$.

If each V_n is convex, so is each $A(r)$, and (9) implies that the same is true of each $B_\delta(0)$, hence also of each translate of $B_\delta(0)$.

We turn to the proof of (6). If $r + s \geq 1$, then $A(r + s) = X$ and (6) is obvious. We may therefore assume that $r + s < 1$, and we will use the following simple proposition about addition in the binary system of notation:

If r, s, and $r + s$ are in D and $c_n(r) + c_n(s) \neq c_n(r + s)$ for some n, then at the smallest n where this happens we have $c_n(r) = c_n(s) = 0$, $c_n(r + s) = 1$.

Put $\alpha_n = c_n(r)$, $\beta_n = c_n(s)$, $\gamma_n = c_n(r + s)$. If $\alpha_n + \beta_n = \gamma_n$ for all n then (3) shows that $A(r) + A(s) = A(r + s)$. In the other case, let N be the smallest integer for which $\alpha_N + \beta_N \neq \gamma_N$. Then, as mentioned above, $\alpha_N = \beta_N = 0$, $\gamma_N = 1$. Hence

$$\begin{aligned} A(r) &\subset \alpha_1 V_1 + \cdots + \alpha_{N-1} V_{N-1} + V_{N+1} + V_{N+2} + \cdots \\ &\subset \alpha_1 V_1 + \cdots + \alpha_{N-1} V_{N-1} + V_{N+1} + V_{N+1}. \end{aligned}$$

Likewise

$$A(s) \subset \beta_1 V_1 + \cdots + \beta_{N-1} V_{N-1} + V_{N+1} + V_{N+1}.$$

Since $\alpha_n + \beta_n = \gamma_n$ for all $n < N$, (1) now leads to

$$A(r) + A(s) \subset \gamma_1 V_1 + \cdots + \gamma_{N-1} V_{n-1} + V_N \subset A(r + s)$$

because $\gamma_N = 1$. ////

1.25 Cauchy sequences (*a*) Suppose d is a metric on a set X. A sequence $\{x_n\}$ in X is a *Cauchy sequence* if to every $\varepsilon > 0$ there corresponds an integer N such that $d(x_m, x_n) < \varepsilon$ whenever $m > N$ and $n > N$. If every Cauchy sequence in X converges to a point of X, then d is said to be a *complete* metric on X.

(*b*) Let τ be the topology of a topological vector space X. The notion of Cauchy sequence can be defined in this setting without reference to any metric: Fix a local base $\mathscr{B}$ for τ. A sequence $\{x_n\}$ in X is then said to be a *Cauchy sequence* if to every $V \in \mathscr{B}$ corresponds an N such that $x_n - x_m \in V$ if $n > N$ and $m > N$.

It is clear that different local bases for the same τ give rise to the same class of Cauchy sequences.

(*c*) Suppose now that X is a topological vector space whose topology τ is compatible with an *invariant* metric d. Let us temporarily use the terms d-Cauchy sequence and τ-Cauchy sequence for the concepts defined in (*a*) and (*b*), respectively. Since

$$d(x_n, x_m) = d(x_n - x_m, 0),$$

and since the d-balls centered at the origin form a local base for τ, we conclude:

A sequence $\{x_n\}$ in X is a d-Cauchy sequence if and only if it is a τ-Cauchy sequence.

Consequently, any two invariant metrics on X that are compatible with τ have the same Cauchy sequences. They clearly also have the same convergent sequences (namely, the τ-convergent ones). These remarks prove the following fact:

If d_1 and d_2 are invariant metrics on a vector space X which induce the same topology on X, then

(a) d_1 and d_2 have the same Cauchy sequences, and
(b) d_1 is complete if and only if d_2 is complete.

Invariance is needed in the hypothesis (Exercise 12).
The following "dilation principle" will be used several times.

1.26 Theorem *Suppose that (X, d_1) and (Y, d_2) are metric spaces, and (X, d_1) is complete. If E is a closed set in X, $f: E \to Y$ is continuous, and*

$$d_2(f(x'), f(x'')) \geq d_1(x', x'')$$

for all $x', x'' \in E$, then $f(E)$ is closed.

PROOF. Pick $y \in \overline{f(E)}$. There exist points $x_n \in E$ so that $y = \lim f(x_n)$. Thus $\{f(x_n)\}$ is Cauchy in Y. Our hypothesis implies therefore that $\{x_n\}$ is Cauchy in X. Being a closed subset of a complete metric space, E is complete; hence there exists $x = \lim x_n$ in E. Since f is continuous,

$$f(x) = \lim f(x_n) = y.$$

Thus $y \in f(E)$. ////

1.27 Theorem *Suppose Y is a subspace of a topological vector space X, and Y is an F-space (in the topology inherited from X). Then Y is a closed subspace of X.*

PROOF. Choose an invariant metric d on Y, compatible with its topology. Let

$$B_{1/n} = \left\{y \in Y: d(y, 0) < \frac{1}{n}\right\},$$

let U_n be a neighborhood of 0 in X such that $Y \cap U_n = B_{1/n}$, and choose symmetric neighborhoods V_n of 0 in X such that $V_n + V_n \subset U_n$ and $V_{n+1} \subset V_n$.

Suppose $x \in \bar{Y}$, and define

$$E_n = Y \cap (x + V_n) \qquad (n = 1, 2, 3, \ldots).$$

If $y_1 \in E_n$ and $y_2 \in E_n$, then $y_1 - y_2$ lies in Y and also in $V_n + V_n \subset U_n$, hence in $B_{1/n}$. The diameters of the sets E_n therefore tend to 0. Since each E_n is nonempty and since Y is complete, it follows that the Y-closures of the sets E_n have exactly one point y_0 in common.

Let W be a neighborhood of 0 in X, and define

$$F_n = Y \cap (x + W \cap V_n).$$

The preceding argument shows that the Y-closures of the sets F_n have one common point y_W. But $F_n \subset E_n$. Hence $y_W = y_0$. Since $F_n \subset x + W$, it follows that y_0 lies in the X-closure of $x + W$, for every W. This implies $y_0 = x$. Thus $x \in Y$. This proves that $\bar{Y} = Y$. ////

The following simple facts are sometimes useful.

1.28 Theorem

(*a*) *If d is a translation-invariant metric on a vector space X then*

$$d(nx, 0) \le nd(x, 0)$$

for every $x \in X$ and for $n = 1, 2, 3, \ldots$.

(*b*) *If $\{x_n\}$ is a sequence in a metrizable topological vector space X and if $x_n \to 0$ as $n \to \infty$, then there are positive scalars γ_n such that $\gamma_n \to \infty$ and $\gamma_n x_n \to 0$.*

PROOF. Statement (*a*) follows from

$$d(nx, 0) \le \sum_{k=1}^{n} d(kx, (k-1)x) = nd(x, 0).$$

To prove (*b*), let d be a metric as in (*a*), compatible with the topology of X. Since $d(x_n, 0) \to 0$, there is an increasing sequence of positive integers n_k such that $d(x_n, 0) < k^{-2}$ if $n \ge n_k$. Put $\gamma_n = 1$ if $n < n_1$; put $\gamma_n = k$ if $n_k \le n < n_{k+1}$. For such n,

$$d(\gamma_n x_n, 0) = d(kx_n, 0) \le kd(x_n, 0) < k^{-1}.$$

Hence $\gamma_n x_n \to 0$ as $n \to \infty$. ////

Boundedness and Continuity

1.29 Bounded sets The notion of a *bounded subset of a topological vector space* X was defined in Section 1.6 and has been encountered several times since then. When X is metrizable, there is a possibility of misunderstanding, since another very familiar notion of boundedness exists in metric spaces.

If d is a metric on a set X, a set $E \subset X$ is said to be d-bounded if there is a number $M < \infty$ such that $d(z, y) \leq M$ for all x and y in E.

If X is a topological vector space with a compatible metric d, the bounded sets and the d-bounded ones need not be the same, even if d is invariant. For instance, if d is a metric such as the one constructed in Theorem 1.24, then X itself is d-bounded (with $M = 1$) but, as we shall see presently, X cannot be bounded, unless $X = \{0\}$. If X is a normed space and d is the metric induced by the norm, then the two notions of boundedness coincide; but if d is replaced by $d_1 = d/(1 + d)$ (an invariant metric which induces the same topology) they do not.

Whenever bounded subsets of a topological vector space are discussed, *it will be understood that the definition is as in Section* 1.6: A set E is bounded if, for every neighborhood V of 0, we have $E \subset tV$ for all sufficiently large t.

We already saw (Theorem 1.15) that *compact sets are bounded.* To see another type of example, let us prove that *Cauchy sequences are bounded* (hence *convergent sequences are bounded*): If $\{x_n\}$ is a Cauchy sequence in X, and V and W are balanced neighborhoods of 0 with $V + V \subset W$, then [part (*b*) of Section 1.25] there exists N such that $x_n \in x_N + V$ for all $n \geq N$. Take $s > 1$ so that $x_N \in sV$. Then

$$x_n \in sV + V \subset sV + sV \subset sW \qquad (n \geq N).$$

Hence $x_n \in tW$ for all $n \geq 1$, if t is sufficiently large.

Also, closures of bounded sets are bounded (Theorem 1.13).

On the other hand, if $x \neq 0$ and $E = \{nx \colon n = 1, 2, 3, \ldots\}$, then E is not bounded, because there is a neighborhood V of 0 that does not contain x; hence nx is not in nV; it follows that no nV contains E.

Consequently, *no subspace of* X (*other than* $\{0\}$) *can be bounded.*

The next theorem characterizes boundedness in terms of sequences.

1.30 Theorem *The following two properties of a set E in a topological vector space are equivalent:*

(*a*) *E is bounded.*

(*b*) *If $\{x_n\}$ is a sequence in E and $\{\alpha_n\}$ is a sequence of scalars such that $\alpha_n \to 0$ as $n \to \infty$, then $\alpha_n x_n \to 0$ as $n \to \infty$.*

PROOF. Suppose E is bounded. Let V be a balanced neighborhood of 0 in X. Then $E \subset tV$ for some t. If $x_n \in E$ and $\alpha_n \to 0$, there exists N such that $|\alpha_n| t < 1$ if $n > N$. Since $t^{-1}E \subset V$ and V is balanced, $\alpha_n x_n \in V$ for all $n > N$. Thus $\alpha_n x_n \to 0$.

Conversely, if E is not bounded, there is a neighborhood V of 0 and a sequence $r_n \to \infty$ such that no $r_n V$ contains E. Choose $x_n \in E$ such that $x_n \notin r_n V$. Then no $r_n^{-1} x_n$ is in V, so that $\{r_n^{-1} x_n\}$ does not converge to 0. ////

1.31 Bounded linear transformations Suppose X and Y are topological vector spaces and $\Lambda: X \to Y$ is linear. Λ is said to be *bounded* if Λ maps bounded sets into bounded sets, i.e., if $\Lambda(E)$ is a bounded subset of Y for every bounded set $E \subset X$.

This definition conflicts with the usual notion of a bounded function as being one whose range is a bounded set. In that sense, no linear function (other than 0) could ever be bounded. Thus when bounded linear mappings (or transformations) are discussed, it is to be understood that the definition is in terms of bounded sets, as above.

1.32 Theorem *Suppose X and Y are topological vector spaces and $\Lambda: X \to Y$ is linear. Among the following four properties of Λ, the implications*

$$(a) \to (b) \to (c)$$

hold. If X is metrizable, then also

$$(c) \to (d) \to (a),$$

so that all four properties are equivalent.

(*a*) *Λ is continuous.*
(*b*) *Λ is bounded.*
(*c*) *If $x_n \to 0$ then $\{\Lambda x_n : n = 1, 2, 3, \ldots\}$ is bounded.*
(*d*) *If $x_n \to 0$ then $\Lambda x_n \to 0$.*

Exercise 13 contains an example in which (*b*) holds but (*a*) does not.

PROOF. Assume (*a*), let E be a bounded set in X, and let W be a neighborhood of 0 in Y. Since Λ is continuous (and $\Lambda 0 = 0$) there is a neighborhood V of 0 in X such that $\Lambda(V) \subset W$. Since E is bounded,

$E \subset tV$ for all large t, so that

$$\Lambda(E) \subset \Lambda(tV) = t\Lambda(V) \subset tW.$$

This shows that $\Lambda(E)$ is a bounded set in Y.

Thus $(a) \to (b)$. Since convergent sequences are bounded, $(b) \to (c)$.

Assume now that X is metrizable, that Λ satisfies (c), and that $x_n \to 0$. By Theorem 1.28, there are positive scalars $\gamma_n \to \infty$ such that $\gamma_n x_n \to 0$. Hence $\{\Lambda(\gamma_n x_n)\}$ is a bounded set in Y, and now Theorem 1.30 implies that

$$\Lambda x_n = \gamma_n^{-1} \Lambda(\gamma_n x_n) \to 0 \qquad \text{as} \quad n \to \infty.$$

Finally, assume that (a) fails. Then there is a neighborhood W of 0 in Y such that $\Lambda^{-1}(W)$ contains no neighborhood of 0 in X. If X has a countable local base, there is therefore a sequence $\{x_n\}$ in X so that $x_n \to 0$ but $\Lambda x_n \notin W$. Thus (d) fails. ////

Seminorms and Local Convexity

1.33 Definitions A *seminorm* on a vector space X is a real-valued function p on X such that

(a) $p(x + y) \leq p(x) + p(y)$ and
(b) $p(\alpha x) = |\alpha| p(x)$

for all x and y in X and all scalars α.

Property (a) is called *subadditivity*. Theorem 1.34 will show that a seminorm p is a norm if it satisfies

(c) $p(x) \neq 0$ if $x \neq 0$.

A family $\mathscr{P}$ of seminorms on X is said to be *separating* if to each $x \neq 0$ corresponds at least one $p \in \mathscr{P}$ with $p(x) \neq 0$.

Next, consider a convex set $A \subset X$ which is *absorbing*, in the sense that every $x \in X$ lies in tA for some $t = t(x) > 0$. [For example, (a) of Theorem 1.15 implies that every neighborhood of 0 in a topological vector space is absorbing. Every absorbing set obviously contains 0.] The *Minkowski functional* μ_A of A is defined by

$$\mu_A(x) = \inf \{t > 0 : t^{-1}x \in A\} \qquad (x \in X).$$

Note that $\mu_A(x) < \infty$ for all $x \in X$, since A is absorbing. The seminorms on X will turn out to be precisely the Minkowski functionals of *balanced* convex absorbing sets.

Seminorms are closely related to local convexity, in two ways: In every locally convex space there exists a separating family of *continuous* seminorms. Conversely, if $\mathscr{P}$ is a separating family of seminorms on a vector space X, then $\mathscr{P}$ can be used to define a locally convex topology on X with the property that every $p \in \mathscr{P}$ is continuous. This is a frequently used method of introducing a topology. The details are contained in Theorems 1.36 and 1.37.

1.34 Theorem *Suppose p is a seminorm on a vector space X. Then*

(*a*) $p(0) = 0$.
(*b*) $|p(x) - p(y)| \le p(x - y)$.
(*c*) $p(x) \ge 0$.
(*d*) $\{x: p(x) = 0\}$ *is a subspace of* X.
(*e*) *The set* $B = \{x: p(x) < 1\}$ *is convex, balanced, absorbing, and* $p = \mu_B$.

PROOF. Statement (*a*) follows from $p(\alpha x) = |\alpha| p(x)$, with $\alpha = 0$. The subadditivity of p shows that

$$p(x) = p(x - y + y) \le p(x - y) + p(y)$$

so that $p(x) - p(y) \le p(x - y)$. This also holds with x and y interchanged. Since $p(x - y) = p(y - x)$, (*b*) follows. With $y = 0$, (*b*) implies (*c*). If $p(x) = p(y) = 0$ and α, β are scalars, (*c*) implies

$$0 \le p(\alpha x + \beta y) \le |\alpha| p(x) + |\beta| p(y) = 0.$$

This proves (*d*).

As to (*e*), it is clear that B is balanced. If $x \in B$, $y \in B$, and $0 < t < 1$, then

$$p(tx + (1 - t)y) \le tp(x) + (1 - t)p(y) < 1.$$

Thus B is convex. If $x \in X$ and $s > p(x)$ then $p(s^{-1}x) = s^{-1}p(x) < 1$. This shows that B is absorbing and also that $\mu_B(x) \le s$. Hence $\mu_B \le p$. But if $0 < t \le p(x)$ then $p(t^{-1}x) \ge 1$, and so $t^{-1}x$ is not in B. This implies $p(x) \le \mu_B(x)$ and completes the proof. ////

1.35 Theorem *Suppose A is a convex absorbing set in a vector space X. Then*

(*a*) $\mu_A(x + y) \le \mu_A(x) + \mu_A(y)$.
(*b*) $\mu_A(tx) = t\mu_A(x)$ *if* $t \ge 0$.

(*c*) μ_A *is a seminorm if A is balanced.*

(*d*) *If* $B = \{x: \mu_A(x) < 1\}$ *and* $C = \{x: \mu_A(x) \le 1\}$, *then* $B \subset A \subset C$ *and* $\mu_B = \mu_A = \mu_C$.

PROOF. If $t = \mu_A(x) + \varepsilon$ and $s = \mu_A(y) + \varepsilon$, for some $\varepsilon > 0$, then x/t and y/s are in A; hence so is their convex combination

$$\frac{x+y}{s+t} = \frac{t}{s+t} \cdot \frac{x}{t} + \frac{s}{s+t} \cdot \frac{y}{s}.$$

This shows that $\mu_A(x + y) \le s + t = \mu_A(x) + \mu_A(y) + 2\varepsilon$, and (*a*) is proved.

Property (*b*) is clear, and (*c*) follows from (*a*) and (*b*).

When we turn to (*d*), the inclusions $B \subset A \subset C$ show that $\mu_C \le \mu_A \le \mu_B$. To prove equality, fix $x \in X$, and choose s, t so that $\mu_C(x) < s < t$. Then $x/s \in C$, $\mu_A(x/s) \le 1$, $\mu_A(x/t) \le s/t < 1$; hence $x/t \in B$, so that $\mu_B(x) \le t$. This holds for every $t > \mu_C(x)$. Hence $\mu_B(x) \le \mu_C(x)$. ////

1.36 Theorem *Suppose* $\mathscr{B}$ *is a convex balanced local base in a topological vector space X. Associate to every* $V \in \mathscr{B}$ *its Minkowski functional* μ_V. *Then*

(*a*) $V = \{x \in X: \mu_V(x) < 1\}$, *for every* $V \in \mathscr{B}$, *and*

(*b*) $\{\mu_V: V \in \mathscr{B}\}$ *is a separating family of continuous seminorms on X.*

PROOF. If $x \in V$, then $x/t \in V$ for some $t < 1$, because V is open; hence $\mu_V(x) < 1$. If $x \notin V$, then $x/t \in V$ implies $t \ge 1$, because V is balanced; hence $\mu_V(x) \ge 1$. This proves (*a*).

Theorem 1.35 shows that each μ_V is a seminorm. If $r > 0$, it follows from (*a*) and Theorem 1.34 that

$$|\mu_V(x) - \mu_V(y)| \le \mu_V(x - y) < r$$

if $x - y \in rV$. Hence μ_V is continuous. If $x \in X$ and $x \neq 0$, then $x \notin V$ for some $V \in \mathscr{B}$. For this V, $\mu_V(x) \ge 1$. Thus $\{\mu_V\}$ is separating. ////

1.37 Theorem *Suppose* $\mathscr{P}$ *is a separating family of seminorms on a vector space X. Associate to each* $p \in \mathscr{P}$ *and to each positive integer n the set*

$$V(p, n) = \left\{x: p(x) < \frac{1}{n}\right\}.$$

Let $\mathscr{B}$ *be the collection of all finite intersections of the sets* $V(p, n)$. *Then* $\mathscr{B}$ *is a convex balanced local base for a topology* τ *on X, which turns X into a locally convex space such that*

Seminorms are closely related to local convexity, in two ways: In every locally convex space there exists a separating family of *continuous* seminorms. Conversely, if $\mathscr{P}$ is a separating family of seminorms on a vector space X, then $\mathscr{P}$ can be used to define a locally convex topology on X with the property that every $p \in \mathscr{P}$ is continuous. This is a frequently used method of introducing a topology. The details are contained in Theorems 1.36 and 1.37.

1.34 Theorem *Suppose p is a seminorm on a vector space X. Then*

(*a*) $p(0) = 0$.
(*b*) $|p(x) - p(y)| \leq p(x - y)$.
(*c*) $p(x) \geq 0$.
(*d*) $\{x: p(x) = 0\}$ *is a subspace of* X.
(*e*) *The set* $B = \{x: p(x) < 1\}$ *is convex, balanced, absorbing, and* $p = \mu_B$.

PROOF. Statement (*a*) follows from $p(\alpha x) = |\alpha| p(x)$, with $\alpha = 0$. The subadditivity of p shows that

$$p(x) = p(x - y + y) \leq p(x - y) + p(y)$$

so that $p(x) - p(y) \leq p(x - y)$. This also holds with x and y interchanged. Since $p(x - y) = p(y - x)$, (*b*) follows. With $y = 0$, (*b*) implies (*c*). If $p(x) = p(y) = 0$ and α, β are scalars, (*c*) implies

$$0 \leq p(\alpha x + \beta y) \leq |\alpha| p(x) + |\beta| p(y) = 0.$$

This proves (*d*).

As to (*e*), it is clear that B is balanced. If $x \in B$, $y \in B$, and $0 < t < 1$, then

$$p(tx + (1 - t)y) \leq tp(x) + (1 - t)p(y) < 1.$$

Thus B is convex. If $x \in X$ and $s > p(x)$ then $p(s^{-1}x) = s^{-1}p(x) < 1$. This shows that B is absorbing and also that $\mu_B(x) \leq s$. Hence $\mu_B \leq p$. But if $0 < t \leq p(x)$ then $p(t^{-1}x) \geq 1$, and so $t^{-1}x$ is not in B. This implies $p(x) \leq \mu_B(x)$ and completes the proof. ////

1.35 Theorem *Suppose A is a convex absorbing set in a vector space X. Then*

(*a*) $\mu_A(x + y) \leq \mu_A(x) + \mu_A(y)$.
(*b*) $\mu_A(tx) = t\mu_A(x)$ *if* $t \geq 0$.

(*c*) *μ_A is a seminorm if A is balanced.*

(*d*) *If $B = \{x\colon \mu_A(x) < 1\}$ and $C = \{x\colon \mu_A(x) \le 1\}$, then $B \subset A \subset C$ and $\mu_B = \mu_A = \mu_C$.*

PROOF. If $t = \mu_A(x) + \varepsilon$ and $s = \mu_A(y) + \varepsilon$, for some $\varepsilon > 0$, then x/t and y/s are in A; hence so is their convex combination

$$\frac{x+y}{s+t} = \frac{t}{s+t} \cdot \frac{x}{t} + \frac{s}{s+t} \cdot \frac{y}{s}.$$

This shows that $\mu_A(x + y) \le s + t = \mu_A(x) + \mu_A(y) + 2\varepsilon$, and (*a*) is proved.

Property (*b*) is clear, and (*c*) follows from (*a*) and (*b*).

When we turn to (*d*), the inclusions $B \subset A \subset C$ show that $\mu_C \le \mu_A \le \mu_B$. To prove equality, fix $x \in X$, and choose s, t so that $\mu_C(x) < s < t$. Then $x/s \in C$, $\mu_A(x/s) \le 1$, $\mu_A(x/t) \le s/t < 1$; hence $x/t \in B$, so that $\mu_B(x) \le t$. This holds for every $t > \mu_C(x)$. Hence $\mu_B(x) \le \mu_C(x)$. ////

1.36 Theorem *Suppose $\mathscr{B}$ is a convex balanced local base in a topological vector space X. Associate to every $V \in \mathscr{B}$ its Minkowski functional μ_V. Then*

(*a*) *$V = \{x \in X\colon \mu_V(x) < 1\}$, for every $V \in \mathscr{B}$, and*

(*b*) *$\{\mu_V\colon V \in \mathscr{B}\}$ is a separating family of continuous seminorms on X.*

PROOF. If $x \in V$, then $x/t \in V$ for some $t < 1$, because V is open; hence $\mu_V(x) < 1$. If $x \notin V$, then $x/t \in V$ implies $t \ge 1$, because V is balanced; hence $\mu_V(x) \ge 1$. This proves (*a*).

Theorem 1.35 shows that each μ_V is a seminorm. If $r > 0$, it follows from (*a*) and Theorem 1.34 that

$$|\mu_V(x) - \mu_V(y)| \le \mu_V(x - y) < r$$

if $x - y \in rV$. Hence μ_V is continuous. If $x \in X$ and $x \ne 0$, then $x \notin V$ for some $V \in \mathscr{B}$. For this V, $\mu_V(x) \ge 1$. Thus $\{\mu_V\}$ is separating. ////

1.37 Theorem *Suppose $\mathscr{P}$ is a separating family of seminorms on a vector space X. Associate to each $p \in \mathscr{P}$ and to each positive integer n the set*

$$V(p, n) = \left\{x\colon p(x) < \frac{1}{n}\right\}.$$

Let $\mathscr{B}$ be the collection of all finite intersections of the sets $V(p, n)$. Then $\mathscr{B}$ is a convex balanced local base for a topology τ on X, which turns X into a locally convex space such that (*a*) *every* $p \in \mathscr{P}$ *is continuous, and*

(*b*) *a set* $E \subset X$ *is bounded if and only if every* $p \in \mathscr{P}$ *is bounded on* E.

PROOF. Declare a set $A \subset X$ to be open if and only if A is a (possibly empty) union of translates of members of $\mathscr{B}$. This clearly defines a translation-invariant topology τ on X; each member of $\mathscr{B}$ is convex and balanced, and $\mathscr{B}$ is a local base for τ.

Suppose $x \in X$, $x \neq 0$. Then $p(x) > 0$ for some $p \in \mathscr{P}$. Since x is not in $V(p, n)$ if $np(x) > 1$, we see that 0 is not in the neighborhood $x - V(p, n)$ of x, so that x is not in the closure of $\{0\}$. Thus $\{0\}$ is a closed set, and since τ is translation-invariant, every point of X is a closed set.

Next we show that addition and scalar multiplication are continuous. Let U be a neighborhood of 0 in X. Then

(1)
$$U \supset V(p_1, n_1) \cap \cdots \cap V(p_m, n_m)$$

for some $p_1, \ldots, p_m \in \mathscr{P}$ and some positive integers $n_1, \ldots, n_m$. Put

(2)
$$V = V(p_1, 2n_1) \cap \cdots \cap V(p_m, 2n_m).$$

Since every $p \in \mathscr{P}$ is subadditive, $V + V \subset U$. This proves that addition is continuous.

Suppose now that $x \in X$, α is a scalar, and U and V are as above. Then $x \in sV$ for some $s > 0$. Put $t = s/(1 + |\alpha| s)$. If $y \in x + tV$ and $|\beta - \alpha| < 1/s$, then

$$\beta y - \alpha x = \beta(y - x) + (\beta - \alpha)x$$

which lies in

$$|\beta| tV + |\beta - \alpha| sV \subset V + V \subset U$$

since $|\beta| t \leq 1$ and V is balanced. This proves that scalar multiplication is continuous.

Thus X is a locally convex space. The definition of $V(p, n)$ shows that every $p \in \mathscr{P}$ is continuous at 0. Hence p is continuous on X, by (*b*) of Theorem 1.34.

Finally, suppose $E \subset X$ is bounded. Fix $p \in \mathscr{P}$. Since $V(p, 1)$ is a neighborhood of 0, $E \subset kV(p, 1)$ for some $k < \infty$. Hence $p(x) < k$ for every $x \in E$. It follows that every $p \in \mathscr{P}$ is bounded on E.

Conversely, suppose E satisfies this condition, U is a neighborhood of 0, and (1) holds. There are numbers $M_i < \infty$ such that $p_i < M_i$ on E $(1 \leq i \leq m)$. If $n > M_i n_i$ for $1 \leq i \leq m$, it follows that $E \subset nU$, so that E is bounded. ////

1.38 Remarks (*a*) It was necessary to take finite intersections of the sets $V(p, n)$ in Theorem 1.37; the sets $V(p, n)$ themselves need not form a local base. (They do form what is usually called a *subbase* for the constructed topology.) To see an example of this, take $X = R^2$, and let $\mathscr{P}$ consist of the seminorms p_1 and p_2 defined by $p_i(x) = |x_i|$; here $x = (x_1, x_2)$. Exercise 8 develops this comment further.

(*b*) Theorems 1.36 and 1.37 raise a natural problem: If $\mathscr{B}$ is a convex balanced local base for the topology τ of a locally convex space X, then $\mathscr{B}$ generates a separating family $\mathscr{P}$ of continuous seminorms on X, as in Theorem 1.36. This $\mathscr{P}$ in turn induces a topology τ_1 on X, by the process described in Theorem 1.37. Is $\tau = \tau_1$?

The answer is affirmative. To see this, note that every $p \in \mathscr{P}$ is τ-continuous, so that the sets $V(p, n)$ of Theorem 1.37 are in τ. Hence $\tau_1 \subset \tau$. Conversely, if $W \in \mathscr{B}$ and $p = \mu_W$, then

$$W = \{x: \mu_W(x) < 1\} = V(p, 1).$$

Thus $W \in \tau_1$ for every $W \in \mathscr{B}$; this implies that $\tau \subset \tau_1$.

(*c*) If $\mathscr{P} = \{p_i: i = 1, 2, 3, \ldots\}$ is a countable separating family of seminorms on X, Theorem 1.37 shows that $\mathscr{P}$ induces a topology τ with a countable local base. By Theorem 1.24, τ is metrizable. In the present situation, a compatible translation-invariant metric can be defined directly in terms of $\{p_i\}$ by setting

$$d(x, y) = \max_i \frac{c_i p_i(x - y)}{1 + p_i(x - y)}, \tag{1}$$

where $\{c_i\}$ is some fixed sequence of positive numbers which converges to 0 as $i \to \infty$.

It is easy to verify that d is a metric on X.

We claim that the balls

$$B_r = \{x: d(0, x) < r\} \qquad (0 < r < \infty) \tag{2}$$

form a convex balanced local base for τ.

Fix r. If $c_i \le r$ (which holds for all but finitely many i, since $c_i \to 0$), then $c_i p_i/(1 + p_i) < r$. Hence B_r is the intersection of *finitely* many sets of the form

$$\left\{x: p_i(x) < \frac{r}{c_i - r}\right\}, \tag{3}$$

namely those for which $c_i > r$. These sets are open, since each p_i is continuous (Theorem 1.37). Thus B_r is open, and, by Theorem 1.34, is also convex and balanced.

Next, let W be a neighborhood of 0 in X. The definition of τ shows that W contains the intersection of appropriately chosen sets

$$V(p_i, \delta_i) = \{x: p_i(x) < \delta_i < 1\} \qquad (1 \le i \le k). \tag{4}$$

If $2r < \min\{c_1\delta_1, \ldots, c_k\delta_k\}$ and $x \in B_r$, then

$$\frac{c_i p_i(x)}{1 + p_i(x)} < r < \frac{c_i \delta_i}{2} \qquad (1 \le i \le k), \tag{5}$$

which implies $p_i(x) < \delta_i$. Thus $B_r \subset W$.

This proves our claim and also shows that d is compatible with τ.

1.39 Theorem *A topological vector space X is normable if and only if its origin has a convex bounded neighborhood.*

PROOF. If X is normable, and if $\|\cdot\|$ is a norm that is compatible with the topology of X, then the open unit ball $\{x: \|x\| < 1\}$ is convex and bounded.

For the converse, assume V is a convex bounded neighborhood of 0. By Theorem 1.14, V contains a convex balanced neighborhood U of 0; of course, U is also bounded. Define

$$\|x\| = \mu(x) \qquad (x \in X) \tag{1}$$

where μ is the Minkowski functional of U.

By (*c*) of Theorem 1.15, the sets rU $(r > 0)$ form a local base for the topology of X. If $x \neq 0$, then $x \notin rU$ for some $r > 0$; hence $\|x\| \geq r$. It now follows from Theorem 1.35 that (1) defines a norm. The definition of the Minkowski functional, together with the fact that U is open, implies that

$$\{x: \|x\| < r\} = rU \tag{2}$$

for every $r > 0$. The norm topology coincides therefore with the given one. ////

Quotient Spaces

1.40 Definitions Let N be a subspace of a vector space X. For every $x \in X$, let $\pi(x)$ be the coset of N that contains x; thus

$$\pi(x) = x + N.$$

These cosets are the elements of a vector space X/N, called the *quotient space of X modulo N*, in which addition and scalar multiplication are defined by

$$\pi(x) + \pi(y) = \pi(x + y), \qquad \alpha\pi(x) = \pi(\alpha x). \tag{1}$$

[Note that now $\alpha\pi(x) = N$ when $\alpha = 0$. This differs from the usual notation, as introduced in Section 1.4.] Since N is a vector space, the operations (1) are well defined. This means that if $\pi(x) = \pi(x')$ (that is, $x' - x \in N$) and

$\pi(y) = \pi(y')$ then

$$\pi(x) + \pi(y) = \pi(x') + \pi(y'), \qquad \alpha\pi(x') = \alpha\pi(x). \tag{2}$$

The origin of X/N is $\pi(0) = N$. By (1), π is a linear mapping of X onto X/N with N as its null space; π is often called the *quotient map* of X onto X/N.

Suppose now that τ is a vector topology on X and that N is a *closed* subspace of X. Let τ_N be the collection of all sets $E \subset X/N$ for which $\pi^{-1}(E) \in \tau$. Then τ_N turns out to be a topology on X/N, called the *quotient topology*. Some of its properties are listed in the next theorem. Recall that an *open mapping* is one that maps open sets to open sets.

1.41 Theorem *Let N be a closed subspace of a topological vector space X. Let τ be the topology of X and define τ_N as above.*

(*a*) *τ_N is a vector topology on X/N; the quotient map $\pi: X \to X/N$ is linear, continuous, and open.*
(*b*) *If $\mathscr{B}$ is a local base for τ, then the collection of all sets $\pi(V)$ with $V \in \mathscr{B}$ is a local base for τ_N.*
(*c*) *Each of the following properties of X is inherited by X/N: local convexity, local boundedness, metrizability, normability.*
(*d*) *If X is an F-space, or a Fréchet space, or a Banach space, so is X/N.*

PROOF. Since $\pi^{-1}(A \cap B) = \pi^{-1}(A) \cap \pi^{-1}(B)$ and

$$\pi^{-1}(\bigcup E_\lambda) = \bigcup \pi^{-1}(E_\lambda),$$

τ_N is a topology. A set $F \subset X/N$ is τ_N-closed if and only if $\pi^{-1}(F)$ is τ-closed. In particular, every point of X/N is closed, since

$$\pi^{-1}(\pi(x)) = N + x$$

and N was assumed to be closed.

The continuity of π follows directly from the definition of τ_N. Next, suppose $V \in \tau$. Since

$$\pi^{-1}(\pi(V)) = N + V$$

and $N + V \in \tau$, it follows that $\pi(V) \in \tau_N$. Thus π is an open mapping.

If now W is a neighborhood of 0 in X/N, there is a neighborhood V of 0 in X such that

$$V + V \subset \pi^{-1}(W).$$

Hence $\pi(V) + \pi(V) \subset W$. Since π is open, $\pi(V)$ is a neighborhood of 0 in X/N. Addition is therefore continuous in X/N.

The continuity of scalar multiplication in X/N is proved in the same manner. This establishes (*a*).

It is clear that (*a*) implies (*b*). With the aid of Theorems 1.32, 1.24, and 1.39, it is just as easy to see that (*b*) implies (*c*).

Suppose next that d is an invariant metric on X, compatible with τ. Define ρ by

$$\rho(\pi(x), \pi(y)) = \inf \{d(x - y, z): z \in N\}.$$

This may be interpreted as the distance from $x - y$ to N. We omit the verifications that are now needed to show that ρ is well defined and that it is an invariant metric on X/N. Since

$$\pi(\{x: d(x, 0) < r\}) = \{u: \rho(u, 0) < r\},$$

it follows from (*b*) that ρ is compatible with τ_N.

If X is normed, this definition of ρ specializes to yield what is usually called the *quotient norm* of X/N:

$$\|\pi(x)\| = \inf \{\|x - z\|: z \in N\}.$$

To prove (*d*) we have to show that ρ is a complete metric whenever d is complete.

Suppose $\{u_n\}$ is a Cauchy sequence in X/N, relative to ρ. There is a subsequence $\{u_{n_i}\}$ with $\rho(u_{n_i}, u_{n_{i+1}}) < 2^{-i}$. One can then inductively choose $x_i \in X$ such that $\pi(x_i) = u_{n_i}$ and $d(x_i, x_{i+1}) < 2^{-i}$. If d is complete, the Cauchy sequence $\{x_i\}$ converges to some $x \in X$. The continuity of π implies that $u_{n_i} \to \pi(x)$ as $i \to \infty$. But if a Cauchy sequence has a convergent subsequence then the full sequence must converge. Hence ρ is complete, and so is the proof of Theorem 1.41.

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Here is an easy application of these concepts:

1.42 Theorem *Suppose N and F are subspaces of a topological vector space X, N is closed, and F has finite dimension. Then N + F is closed.*

PROOF. Let π be the quotient map of X onto X/N, and give X/N its quotient topology. Then $\pi(F)$ is a finite-dimensional subspace of X/N; since X/N is a topological vector space, Theorem 1.21 implies that $\pi(F)$ is closed in X/N. Since $N + F = \pi^{-1}(\pi(F))$ and π is continuous, we conclude that $N + F$ is closed. (Compare Exercise 20.) ////

1.43 Seminorms and quotient spaces Suppose p is a seminorm on a vector space X and

$$N = \{x: p(x) = 0\}.$$

Then N is a subspace of X (Theorem 1.34). Let π be the quotient map of X onto X/N, and define

$$\tilde{p}(\pi(x)) = p(x).$$

If $\pi(x) = \pi(y)$, then $p(x - y) = 0$, and since

$$|p(x) - p(y)| \le p(x - y)$$

it follows that $\tilde{p}(\pi(x)) = \tilde{p}(\pi(y))$. Thus $\tilde{p}$ is well defined on X/N, and it is now easy to verify that $\tilde{p}$ is a *norm* on X/N.

Here is a familiar example of this. Fix r, $1 \le r < \infty$; let L^r be the space of all Lebesgue measurable functions on $[0, 1]$ for which

$$p(f) = \|f\|_r = \left\{\int_0^1 |f(t)|^r \, dt\right\}^{1/r} < \infty.$$

This defines a seminorm on L^r, not a norm, since $\|f\|_r = 0$ whenever $f = 0$ almost everywhere. Let N be the set of these "null functions." Then L^r/N is the Banach space that is usually called L^r. The norm of L^r is obtained by the above passage from p to $\tilde{p}$.

Examples

1.44 The spaces $C(\Omega)$ If Ω is a nonempty open set in some euclidean space, then Ω is the union of countably many compact sets $K_n \neq \varnothing$ which can be chosen so that K_n lies in the interior of K_{n+1} ($n = 1, 2, 3, \ldots$). $C(\Omega)$ is the vector space of all complex-valued continuous functions on Ω, topologized by the separating family of seminorms

(1) $$p_n(f) = \sup \{|f(x)| : x \in K_n\},$$

in accordance with Theorem 1.37. Since $p_1 \le p_2 \le \cdots$, the sets

(2) $$V_n = \left\{f \in C(\Omega) : p_n(f) < \frac{1}{n}\right\} \qquad (n = 1, 2, 3, \ldots)$$

form a convex local base for $C(\Omega)$. According to remark (*c*) of Section 1.38, the topology of $C(\Omega)$ is compatible with the metric

(3) $$d(f, g) = \max_n \frac{2^{-n} p_n(f - g)}{1 + p_n(f - g)}.$$

If $\{f_i\}$ is a Cauchy sequence relative to this metric, then $p_n(f_i - f_j) \to 0$ for every n, as $i, j \to \infty$, so that $\{f_i\}$ converges uniformly on K_n, to a function $f \in C(\Omega)$. An easy computation then shows $d(f, f_i) \to 0$. Thus d is a complete metric. We have now proved *that $C(\Omega)$ is a Fréchet space.*

By (*b*) of Theorem 1.37, a set $E \subset C(\Omega)$ is bounded if and only if there are numbers $M_n < \infty$ such that $p_n(f) \le M_n$ for all $f \in E$; explicitly,

(4) $$|f(x)| \le M_n \qquad \text{if } f \in E \text{ and } x \in K_n.$$

Since every V_n contains an f for which $p_{n+1}(f)$ is as large as we please, it follows that no V_n is bounded. *Thus* $C(\Omega)$ *is not locally bounded, hence is not normable.*

1.45 The spaces $H(\Omega)$ Let Ω now be a nonempty open subset of the complex plane, define $C(\Omega)$ as in Section 1.44, and let $H(\Omega)$ be the subspace of $C(\Omega)$ that consists of the holomorphic functions in Ω. Since sequences of holomorphic functions that converge uniformly on compact sets have holomorphic limits, $H(\Omega)$ is a closed subspace of $C(\Omega)$. *Hence* $H(\Omega)$ *is a Fréchet space.*

We shall now prove that $H(\Omega)$ *has the Heine-Borel property*. It will then follow from Theorem 1.23 that $H(\Omega)$ *is not locally bounded, hence is not normable.*

Let E be a closed and bounded subset of $H(\Omega)$. Then E satisfies inequalities such as (4) of Section 1.44. Montel's classical theorem about normal families (Th. 14.6 of [23][1]) implies therefore that every sequence $\{f_i\} \subset E$ has a subsequence that converges uniformly on compact subsets of Ω [hence in the topology of $H(\Omega)$] to some $f \in H(\Omega)$. Since E is closed, $f \in E$. This proves that E is compact.

1.46 The spaces $C^\infty(\Omega)$ and $\mathscr{D}_K$ We begin this section by introducing some terminology that will be used in our later work with distributions.

In any discussion of functions of n variables, the term *multi-index* denotes an ordered n-tuple

$$\alpha = (\alpha_1, \ldots, \alpha_n) \tag{1}$$

of nonnegative integers α_i. With each multi-index α is associated the differential operator

$$D^\alpha = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n} \tag{2}$$

whose *order* is

$$|\alpha| = \alpha_1 + \cdots + \alpha_n. \tag{3}$$

If $|\alpha| = 0$, $D^\alpha f = f$.

A complex function f defined in some nonempty open set $\Omega \subset R^n$ is said to belong to $C^\infty(\Omega)$ if $D^\alpha f \in C(\Omega)$ for every multi-index α.

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[1] Numbers in brackets refer to sources listed in the bibliography.

The *support* of a complex function f (on any topological space) is the closure of $\{x: f(x) \neq 0\}$.

If K is a compact set in R^n, then $\mathscr{D}_K$ denotes the space of all $f \in C^\infty(R^n)$ whose support lies in K. (The letter $\mathscr{D}$ has been used for these spaces ever since Schwartz published his work on distributions.) If $K \subset \Omega$, then $\mathscr{D}_K$ may be identified with a subspace of $C^\infty(\Omega)$.

We now define a topology on $C^\infty(\Omega)$ which makes $C^\infty(\Omega)$ into a Fréchet space with the Heine-Borel property, such that $\mathscr{D}_K$ is a closed subspace of $C^\infty(\Omega)$ whenever $K \subset \Omega$.

To do this, choose compact sets K_i $(i = 1, 2, 3, \ldots)$ such that K_i lies in the interior of K_{i+1} and $\Omega = \bigcup K_i$. Define seminorms p_N on $C^\infty(\Omega)$, $N = 1, 2, 3, \ldots$, by setting

$$p_N(f) = \max \{ |D^\alpha f(x)| : x \in K_N, |\alpha| \leq N \}. \tag{4}$$

They define a metrizable locally convex topology on $C^\infty(\Omega)$; see Theorem 1.37 and remark (*c*) of Section 1.38. For each $x \in \Omega$, the functional $f \to f(x)$ is continuous in this topology. Since $\mathscr{D}_K$ is the intersection of the null spaces of these functionals, as x ranges over the complement of K, it follows that $\mathscr{D}_K$ is closed in $C^\infty(\Omega)$.

A local base is given by the sets

$$V_N = \left\{ f \in C^\infty(\Omega): p_N(f) < \frac{1}{N} \right\} \qquad (N = 1, 2, 3, \ldots). \tag{5}$$

If $\{f_i\}$ is a Cauchy sequence in $C^\infty(\Omega)$ (see Section 1.25) and if N is fixed, then $f_i - f_j \in V_N$ if i and j are sufficiently large. Thus $|D^\alpha f_i - D^\alpha f_j| < 1/N$ on K_N, if $|\alpha| \leq N$. It follows that each $D^\alpha f_i$ converges (uniformly on compact subsets of Ω) to a function g_α. In particular, $f_i(x) \to g_0(x)$. It is now evident that $g_0 \in C^\infty(\Omega)$, that $g_\alpha = D^\alpha g_0$, and that $f_i \to g$ in the topology of $C^\infty(\Omega)$.

Thus $C^\infty(\Omega)$ is a Fréchet space. The same is true of each of its closed subspaces $\mathscr{D}_K$.

Suppose next that $E \subset C^\infty(\Omega)$ is closed and bounded. By Theorem 1.37, the boundedness of E is equivalent to the existence of numbers $M_N < \infty$ such that $p_N(f) \leq M_N$ for $N = 1, 2, 3, \ldots$ and for all $f \in E$. The inequalities $|D^\alpha f| \leq M_N$, valid on K_N when $|\alpha| \leq N$, imply the equicontinuity of $\{D^\beta f : f \in E\}$ on K_{N-1}, if $|\beta| \leq N - 1$. It now follows from Ascoli's theorem (proved in Appendix A) and Cantor's diagonal process that every sequence in E contains a subsequence $\{f_i\}$ for which $\{D^\beta f_i\}$ converges, uniformly on compact subsets of Ω, for each multi-index β. Hence $\{f_i\}$ converges in the topology of $C^\infty(\Omega)$. This proves that E is compact.

Hence $C^\infty(\Omega)$ has the Heine-Borel property. It follows from Theorem 1.23 that $C^\infty(\Omega)$ *is not locally bounded, hence not normable.* The same conclusion holds for $\mathscr{D}_K$ whenever K has nonempty interior (otherwise $\mathscr{D}_K = \{0\}$), because $\dim \mathscr{D}_K = \infty$ in that case. This last statement is a consequence of the following proposition:

If B_1 and B_2 are concentric closed balls in R^n, with B_1 in the interior of B_2, then there exists $\phi \in C^\infty(R^n)$ such that $\phi(x) = 1$ for every $x \in B_1$, $\phi(x) = 0$ for every x outside B_2, and $0 \leq \phi \leq 1$ on R^n.

To find such a ϕ, we construct $g \in C^\infty(R^1)$ such that $g(x) = 0$ for $x < a$, $g(x) = 1$ for $x > b$ (where $0 < a < b < \infty$ are preassigned) and put

$$\phi(x_1, \ldots, x_n) = 1 - g(x_1^2 + \cdots + x_n^2). \tag{6}$$

The following construction of g has the advantage that suitable choices of $\{\delta_i\}$ can lead to functions with other desired properties.

Suppose $0 < a < b < \infty$. Choose positive numbers $\delta_0, \delta_1, \delta_2, \ldots$, with $\Sigma \delta_i = b - a$; put

$$m_n = \frac{2^n}{\delta_1 \cdots \delta_n} \qquad (n = 1, 2, 3, \ldots); \tag{7}$$

let f_0 be a continuous monotonic function such that $f_0(x) = 0$ when $x < a$, $f_0(x) = 1$ when $x > a + \delta_0$; and define

$$f_n(x) = \frac{1}{\delta_n} \int_{x-\delta_n}^{x} f_{n-1}(t)\, dt \qquad (n = 1, 2, 3, \ldots). \tag{8}$$

Differentiation of this integral shows, by induction, that f_n has n continuous derivatives and that $|D^n f_n| \leq m_n$. If $n > r$, then

$$D^r f_n(x) = \frac{1}{\delta_n} \int_0^{\delta_n} (D^r f_{n-1})(x - t)\, dt, \tag{9}$$

so that

$$|D^r f_n| \leq m_r \qquad (n \geq r), \tag{10}$$

again by induction on n. The mean value theorem, applied to (9), shows that

$$|D^r f_n - D^r f_{n-1}| \leq m_{r+1}\, \delta_n \qquad (n \geq r + 2). \tag{11}$$

Since $\Sigma \delta_n < \infty$, each $\{D^r f_n\}$ converges, uniformly on $(-\infty, \infty)$, as $n \to \infty$. Hence $\{f_n\}$ converges to a function g, with $|D^r g| \leq m_r$ for $r = 1, 2, 3, \ldots$, such that $g(x) = 0$ for $x < a$ and $g(x) = 1$ for $x > b$.

1.47 The spaces L^p with $0 < p < 1$ Consider a fixed p in this range. The elements of L^p are those Lebesgue measurable functions f on $[0, 1]$ for which

$$\Delta(f) = \int_0^1 |f(t)|^p\, dt < \infty, \tag{1}$$

with the usual identification of functions that coincide almost everywhere. Since $0 < p < 1$, the inequality

(2) $$(a + b)^p \le a^p + b^p$$

holds when $a \ge 0$ and $b \ge 0$. This gives

(3) $$\Delta(f + g) \le \Delta(f) + \Delta(g),$$

so that

(4) $$d(f, g) = \Delta(f - g)$$

defines an *invariant metric* on L^p. That this d is *complete* is proved in the same way as in the familiar case $p \ge 1$. The balls

(5) $$B_r = \{f \in L^p: \Delta(f) < r\}$$

form a local base for the topology of L^p. Since $B_1 = r^{-1/p}B_r$, for all $r > 0$, B_1 is bounded.

Thus L^p is a locally bounded F-space.

We claim that *L^p contains no convex open sets, other than $\varnothing$ and L^p.*

To prove this, suppose $V \ne \varnothing$ is open and convex in L^p. Assume $0 \in V$, without loss of generality. Then $V \supset B_r$, for some $r > 0$. Pick $f \in L^p$. Since $p < 1$, there is a positive integer n such that $n^{p-1}\,\Delta(f) < r$. By the continuity of the indefinite integral of $|f|^p$, there are points

$$0 = x_0 < x_1 < \cdots < x_n = 1$$

such that

(6) $$\int_{x_{i-1}}^{x_i} |f(t)|^p\,dt = n^{-1}\,\Delta(f) \qquad (1 \le i \le n).$$

Define $g_i(t) = nf(t)$ if $x_{i-1} < t \le x_i$, $g_i(t) = 0$ otherwise. Then $g_i \in V$, since (6) shows

(7) $$\Delta(g_i) = n^{p-1}\,\Delta(f) < r \qquad (1 \le i \le n)$$

and $V \supset B_r$. Since V is convex and

(8) $$f = \frac{1}{n}(g_1 + \cdots + g_n),$$

it follows that $f \in V$. Hence $V = L^p$.

This lack of convex open sets has a curious consequence.

Suppose $\Lambda: L^p \to Y$ is a continuous linear mapping of L^p into some locally convex space Y. Let $\mathscr{B}$ be a convex local base for Y. If $W \in \mathscr{B}$, then $\Lambda^{-1}(W)$ is convex, open, not empty. Hence $\Lambda^{-1}(W) = L^p$. Consequently, $\Lambda(L^p) \subset W$ for every $W \in \mathscr{B}$. We conclude that $\Lambda f = 0$ for every $f \in L^p$.

Thus 0 is the only continuous linear mapping of L^p into any locally convex space Y, if $0 < p < 1$. In particular, 0 is the only continuous linear functional on these L^p-spaces.

This is, of course, in violent contrast to the familiar case $p \ge 1$.

Exercises

1. Suppose X is a vector space. All sets mentioned below are understood to be subsets of X. Prove the following statements from the axioms as given in Section 1.4. (Some of these are tacitly used in the text.)

(*a*) If $x \in X$ and $y \in X$ there is a unique $z \in X$ such that $x + z = y$.
(*b*) $0x = 0 = \alpha 0$ if $x \in X$ and α is a scalar.
(*c*) $2A \subset A + A$; it may happen that $2A \neq A + A$.
(*d*) A is convex if and only if $(s + t)A = sA + tA$ for all positive scalars s and t.
(*e*) Every union (and intersection) of balanced sets is balanced.
(*f*) Every intersection of convex sets is convex.
(*g*) If Γ is a collection of convex sets that is totally ordered by set inclusion, then the union of all members of Γ is convex.
(*h*) If A and B are convex, so is $A + B$.
(*i*) If A and B are balanced, so is $A + B$.
(*j*) Show that parts (*f*), (*g*), and (*h*) hold with subspaces in place of convex sets.

2. The *convex hull* of a set A in a vector space X is the set of all *convex combinations* of members of A, that is, the set of all sums

$$t_1 x_1 + \cdots + t_n x_n$$

in which $x_i \in A$, $t_i \geq 0$, $\sum t_i = 1$; n is arbitrary. Prove that the convex hull of A is convex and that it is the intersection of all convex sets that contain A.

3. Let X be a topological vector space. All sets mentioned below are understood to be the subsets of X. Prove the following statements.

(*a*) The convex hull of every open set is open.
(*b*) If X is locally convex then the convex hull of every bounded set is bounded. (This is false without local convexity; see Section 1.47.)
(*c*) If A and B are bounded, so is $A + B$.
(*d*) If A and B are compact, so is $A + B$.
(*e*) If A is compact and B is closed, then $A + B$ is closed.
(*f*) The sum of two closed sets may fail to be closed. [The inclusion in (*b*) of Theorem 1.13 may therefore be strict.]

4. Let $B = \{(z_1, z_2) \in \mathcal{C}^2 : |z_1| \leq |z_2|\}$. Show that B is balanced but that its interior is not. [Compare with (*e*) of Theorem 1.13.]

5. Consider the definition of "bounded set" given in Section 1.6. Would the content of this definition be altered if it were required merely that to every neighborhood V of 0 corresponds *some* $t > 0$ such that $E \subset tV$?

6. Prove that a set E in a topological vector space is bounded if and only if every countable subset of E is bounded.

7. Let X be the vector space of all complex functions on the unit interval $[0, 1]$, topologized by the family of seminorms

$$p_x(f) = |f(x)| \qquad (0 \leq x \leq 1).$$

This topology is called the *topology of pointwise convergence*. Justify this terminology.

Show that there is a sequence $\{f_n\}$ in X such that (*a*) $\{f_n\}$ converges to 0 as $n \to \infty$, but (*b*) if $\{\gamma_n\}$ is any sequence of scalars such that $\gamma_n \to \infty$ then $\{\gamma_n f_n\}$

does not converge to 0. (Use the fact that the collection of all complex sequences converging to 0 has the same cardinality as [0, 1].)

This shows that metrizability cannot be omitted in (*b*) of Theorem 1.28.

8. (*a*) Suppose $\mathscr{P}$ is a separating family of seminorms on a vector space X. Let $\mathscr{Q}$ be the smallest family of seminorms on X that contains $\mathscr{P}$ and is closed under max. [This means: If $p_1 \in \mathscr{Q}$, $p_2 \in \mathscr{Q}$, and $p = \max(p_1, p_2)$, then $p \in \mathscr{Q}$.] If the construction of Theorem 1.37 is applied to $\mathscr{P}$ and to $\mathscr{Q}$, show that the two resulting topologies coincide. The main difference is that $\mathscr{Q}$ leads directly to a base, rather than to a subbase. [See Remark (*a*) of Section 1.38.]

(*b*) Suppose $\mathscr{Q}$ is as in part (*a*) and Λ is a linear functional on X. Show that Λ is continuous if and only if there exists a $p \in \mathscr{Q}$ such that $|\Lambda x| \leq Mp(x)$ for all $x \in X$ and some constant $M < \infty$.

9. Suppose
(*a*) X and Y are topological vector spaces,
(*b*) $\Lambda: X \to Y$ is linear,
(*c*) N is a closed subspace of X,
(*d*) $\pi: X \to X/N$ is the quotient map, and
(*e*) $\Lambda x = 0$ for every $x \in N$.

Prove that there is a unique $f: X/N \to Y$ which satisfies $\Lambda = f \circ \pi$, that is, $\Lambda x = f(\pi(x))$ for all $x \in X$. Prove that this f is linear and that Λ is continuous if and only if f is continuous. Also, Λ is open if and only if f is open.

10. Suppose X and Y are topological vector spaces, $\dim Y < \infty$, $\Lambda: X \to Y$ is linear, and $\Lambda(X) = Y$.
(*a*) Prove that Λ is an open mapping.
(*b*) Assume, in addition, that the null space of Λ is closed, and prove that Λ is then continuous.

11. If N is a subspace of a vector space X, the *codimension* of N in X is, by definition, the dimension of the quotient space X/N.

Suppose $0 < p < 1$ and prove that every subspace of finite codimension is dense in L^p. (See Section 1.47.)

12. Suppose $d_1(x, y) = |x - y|$, $d_2(x, y) = |\phi(x) - \phi(y)|$, where $\phi(x) = x/(1 + |x|)$. Prove that d_1 and d_2 are metrics on R which induce the same topology, although d_1 is complete and d_2 is not.

13. Let C be the vector space of all complex continuous functions on [0, 1]. Define

$$d(f, g) = \int_0^1 \frac{|f(x) - g(x)|}{1 + |f(x) - g(x)|}\, dx.$$

Let (C, σ) be C with the topology induced by this metric. Let (C, τ) be the topological vector space defined by the seminorms

$$p_x(f) = |f(x)| \qquad (0 \leq x \leq 1),$$

in accordance with Theorem 1.37.

(*a*) Prove that every τ-bounded set in C is also σ-bounded and that the identity map $\text{id}: (C, \tau) \to (C, \sigma)$ therefore carries bounded sets into bounded sets.

(*b*) Prove that $\text{id}: (C, \tau) \to (C, \sigma)$ is nevertheless not continuous, although it is sequentially continuous (by Lebesgue's dominated convergence theorem).

Hence (C, τ) is not metrizable. (See Appendix A6, or Theorem 1.32.) Show also directly that (C, τ) has no countable local base.

(*c*) Prove that every continuous linear functional on (C, τ) is of the form

$$f \to \sum_{i=1}^{n} c_i f(x_i)$$

for some choice of $x_1, \ldots, x_n$ in $[0, 1]$ and some $c_i \in \mathbb{C}$.

(*d*) Prove that (C, σ) contains no convex open sets other than $\varnothing$ and C.

(*e*) Prove that $\mathrm{id}: (C, \sigma) \to (C, \tau)$ is not continuous.

14. Put $K = [0, 1]$ and define $\mathscr{D}_K$ as in Section 1.46. Show that the following three families of seminorms (where $n = 0, 1, 2, \ldots$) define the same topology on $\mathscr{D}_K$, if $D = d/dx$:

(*a*) $\|D^n f\|_\infty = \sup\{|D^n f(x)| : -\infty < x < \infty\}$.

(*b*) $\|D^n f\|_1 = \int_0^1 |D^n f(x)|\, dx$.

(*c*) $\|D^n f\|_2 = \left\{\int_0^1 |D^n f(x)|^2\, dx\right\}^{1/2}$.

15. Prove that the spaces $C(\Omega)$ (Section 1.44) do not have the Heine-Borel property.

16. Prove that the topology of $C(\Omega)$ does not depend on the particular choice of $\{K_n\}$, as long as this sequence satisfies the conditions specified in Section 1.44. Do the same for $C^\infty(\Omega)$ (Section 1.46).

17. In the setting of Section 1.46, prove that $f \to D^\alpha f$ is a continuous mapping of $C^\infty(\Omega)$ into $C^\infty(\Omega)$ and also of $\mathscr{D}_K$ into $\mathscr{D}_K$, for every multi-index α.

18. Prove the proposition concerning addition in the binary system which was used at the end of the proof of Theorem 1.24.

19. Suppose M is a dense subspace of a topological vector space X, Y is an F-space, and $\Lambda: M \to Y$ is continuous (relative to the topology that M inherits from X) and linear. Prove that Λ has a continuous linear extension $\tilde{\Lambda}: X \to Y$.

Suggestion: Let V_n be balanced neighborhoods of 0 in X such that $V_n + V_n \subset V_{n-1}$ and such that $d(0, \Lambda x) < 2^{-n}$ if $x \in M \cap V_n$. If $x \in X$ and $x_n \in (x + V_n) \cap M$, show that $\{\Lambda x_n\}$ is a Cauchy sequence in Y, and define $\tilde{\Lambda}x$ to be its limit. Show that $\tilde{\Lambda}$ is well defined, that $\tilde{\Lambda}x = \Lambda x$ if $x \in M$, and that $\tilde{\Lambda}$ is linear and continuous.

20. For each real number t and each integer n, define $e_n(t) = e^{int}$, and define

$$f_n = e_{-n} + ne_n \qquad (n = 1, 2, 3, \ldots).$$

Regard these functions as members of $L^2(-\pi, \pi)$. Let X_1 be the smallest closed subspace of L^2 that contains $e_0, e_1, e_2, \ldots$, and let X_2 be the smallest closed subspace of L^2 that contains $f_1, f_2, f_3, \ldots$. Show that $X_1 + X_2$ is dense in L^2 but not closed. For instance, the vector

$$x = \sum_{n=1}^{\infty} n^{-1} e_{-n}$$

is in L^2 but not in $X_1 + X_2$. (Compare with Theorem 1.42.)

21. Let V be a neighborhood of 0 in a topological vector space X. Prove that there is a real continuous function f on X such that $f(0) = 0$ and $f(x) = 1$ outside V. (Thus X is a *completely regular* topological space.) *Suggestion:* Let V_n be balanced neighborhoods of 0 such that $V_1 + V_1 \subset V$ and $V_{n+1} + V_{n+1} \subset V_n$. Construct f as in the proof of Theorem 1.24. Show that f is continuous at 0 and that

$$|f(x) - f(y)| \leq f(x - y).$$

22. If f is a complex function defined on the compact interval $I = [0, 1] \subset R$, define

$$\omega_\delta(f) = \sup \{|f(x) - f(y)| : |x - y| \leq \delta, x \in I, y \in I\}.$$

If $0 < \alpha \leq 1$, the corresponding *Lipschitz space* Lip α consists of all f for which

$$\|f\| = |f(0)| + \sup \{\delta^{-\alpha}\omega_\delta(f): \delta > 0\}$$

is finite. Define

$$\text{lip } \alpha = \{f \in \text{Lip } \alpha: \lim_{\delta \to 0} \delta^{-\alpha}\omega_\delta(f) = 0\}.$$

Prove that Lip α is a Banach space and that lip α is a closed subspace of Lip α.

23. Let X be the vector space of all continuous functions on the open segment (0, 1). For $f \in X$ and $r > 0$, let $V(f, r)$ consist of all $g \in X$ such that $|g(x) - f(x)| < r$ for all $x \in (0, 1)$. Let τ be the topology on X that these sets $V(f, r)$ generate. Show that addition is τ-continuous but scalar multiplication is not.

24. Show that the set W that occurs in the proof of Theorem 1.14 need not be convex, and that A need not be balanced unless U is convex.