Chapter C

Properties of Legendre Polynomials

C1 Definitions

The Legendre Polynomials are the everywhere regular solutions of *Legendre's Equation*,

$$(1-x^2)u'' - 2xu' + mu = [(1-x^2)u']' + mu = 0, \tag{C.1}$$

which are possible only if

$$m = n(n+1), \quad n = 0, 1, 2, \cdots. \tag{C.2}$$

We write the solution for a particular value of n as $\mathrm{P}_n(x)$. It is a polynomial of degree n. If n is even/odd then the polynomial is even/odd. They are *normalised* such that $\mathrm{P}_n(1) = 1$.

$$\begin{aligned} \mathrm{P}_0(x) &= 1, \\ \mathrm{P}_1(x) &= x, \\ \mathrm{P}_2(x) &= (3x^2-1)/2, \\ \mathrm{P}_0(x) &= (5x^3-3x)/2. \end{aligned}$$

C2 Rodrigue's Formula

They can also be represented using *Rodrigue's Formula*

$$\mathrm{P}_n(x) = \frac{1}{2^n n!}\frac{\mathrm{d}^n}{\mathrm{d}x^n}(x^2-1)^n. \tag{C.3}$$

This can be demonstrated through the following observations

C2.1 Its a polynomial

The right hand side of (C.3) is a polynomial.

C2.2 It takes the value 1 at 1.

If

$$v(x) = \frac{1}{2^n n!}\frac{\mathrm{d}^n}{\mathrm{d}x^n}(x^2-1)^n,$$

then, treating $(x^2-1)^n = (x-1)^n(x+1)^n$ as a product and using Leibnitz' rule to differentiate n times, we have

$$v(x) = \frac{1}{2^n n!}\left(n!(x+1)^n + \text{terms with } (x-1) \text{ as a factor}\right),$$

so that

$$v(1) = \frac{n! 2^n}{2^n n!} = 1.$$

C2.3 It satisfies the equation

Finally

$$(1-x^2)v'' - 2xv' + n(n+1)v = 0,$$

since, if $h(x) = (1-x^2)^n$, then $h' = -2nx(1-x^2)^{n-1}$, so that

$$(1-x^2)h' + 2nxh = 0.$$

Now differentiate $n+1$ times, using Leibnitz, to get

$$(1-x^2)h^{n+2} - 2(n+1)xh^{n+1} - 2\frac{(n+1)n}{2}h^n + 2nxh^{n+1} + 2n(n+1)h^n = 0,$$

or

$$(1-x^2)h^{n+2} - 2xh^{n+1} + n(n+1)h^n = 0.$$

As the equation is linear and $v \propto h$, v satisfies the equation also.

C2.4 And that's it

Thus $v(x)$ is proportional to the regular solution of Legendre's equation and $v(1) = \mathrm{P}(1) = 1$, so $v(x) = \mathrm{P}(x)$.

C3 Orthogonality of Legendre Polynomials

The differential equation and boundary conditions satisfies by the Legendre Polynomials forms a Sturm-Liouiville system (actually a "generalised system" where the boundary condition amounts to insisting on regularity of the solutions at the boundaries). They should therefore satisfy the *orthogonality* relation

$$\int_{-1}^{1} \mathrm{P}_n(x)\,\mathrm{P}_m(x)\; dx = 0, \quad n \neq m. \tag{C.4}$$

If P_n and P_m are solutions of Legendre's equation then

$$[(1-x^2)\,\mathrm{P}_n']' + n(n+1)\,\mathrm{P}_n = 0, \tag{C.5}$$

$$[(1-x^2)\,\mathrm{P}_m']' + m(m+1)\,\mathrm{P}_m = 0. \tag{C.6}$$

Integrating the combination P_m(C.5)$-$ P_n(C.6) gives

$$\int_{-1}^{1} \mathrm{P}_m[(1-x^2)\,\mathrm{P}_n']' - \mathrm{P}_n[(1-x^2)\,\mathrm{P}_m']'\; \mathrm{d}x + [n(n+1) - m(m+1)]\int_{-1}^{1} \mathrm{P}_n\,\mathrm{P}_m\; \mathrm{d}x = 0.$$

Using integration by parts gives, for the first integral,

$$\left[\mathrm{P}_m(1-x^2)\,\mathrm{P}_n'\right]_{-1}^{1} - \left[\mathrm{P}_n(1-x^2)\,\mathrm{P}_m'\right]_{-1}^{1} - \int_{-1}^{1} \underbrace{\mathrm{P}_m'(1-x^2)\,\mathrm{P}_n' - \mathrm{P}_n'(1-x^2)\,\mathrm{P}_m'}_{=0}\ \mathrm{d}x = 0,$$

as $\mathrm{P}_{m,n}$ and their derivatives are finite at $x=\pm 1$ (i.e. they are regular there). Hence, if $n \neq m$

$$\int_{-1}^{1} \mathrm{P}_n\,\mathrm{P}_m\ \mathrm{d}x = 0. \tag{C.7}$$

C4 What is $\int_{-1}^{1} \mathrm{P}_n^2\ \mathrm{d}x$?

We can evaluate this integral using Rodrigue's formula. We have

$$I_n = \int_{-1}^{1} \mathrm{P}_n^2\ \mathrm{d}x = \frac{1}{2^{2n}(n!)^2}\int_{-1}^{1} \frac{\mathrm{d}^n}{\mathrm{d}x^n}(x^2-1)^n \frac{d^n}{dx^n}(x^2-1)^n\ \mathrm{d}x.$$

Integrating by parts gives

$$(2^{2n}(n!)^2)I_n = \left[\frac{\mathrm{d}^{n-1}}{\mathrm{d}x^{n-1}}(x^2-1)^n\right]_{-1}^{1} - \int_{-1}^{1} \frac{\mathrm{d}^{n-1}}{\mathrm{d}x^{n-1}}(x^2-1)^n \frac{\mathrm{d}^{n+1}}{\mathrm{d}x^{n+1}}(x^2-1)^n\ \mathrm{d}x.$$

Note that differentiating $(x^2-1)^n$ anything less than n times leaves an expression that has (x^2-1) as a factor so that the first of these two terms vanishes. Similarly, integrating by parts n times gives

$$I_n = \frac{(-1)^n}{2^{2n}(n!)^2}\int_{-1}^{1}(x^2-1)^n \frac{\mathrm{d}^{2n}}{\mathrm{d}x^{2n}}(x^2-1)^n\ \mathrm{d}x.$$

The $(2n)$th derivative of the polynomial $(x^2-1)^n$, which has degree $2n$ is $(2n)!$. Thus

$$I_n = \frac{(-1)^n(2n)!}{2^{2n}(n!)^2}\int_{-1}^{1}(x^2-1)^n\ \mathrm{d}x.$$

The completion of this argument is left as an exercise. One way to proceed is to use the transformation $s=(x+1)/2$ to transform the integral and then use a reduction formula to show that

$$\int_0^1 s^n(1-s)^n\ \mathrm{d}s = \frac{(n!)^2}{(2n+1)!}.$$

The final result is

$$\int_{-1}^{1} \mathrm{P}_n^2\ \mathrm{d}x = \frac{2}{2n+1}. \tag{C.8}$$

C5 Generalised Fourier Series

Sturm-Liouiville theory does more than guarantee the orthogonality of Legendre polynomials, it also shows that we can represent functions on $[-1,1]$ as a sum of Legendre Polynomials. Thus for suitable $f(x)$ on $[-1,1]$ we have the *generalized Fourier series*

$$f(x) = \sum_0^\infty a_n\,\mathrm{P}_n(x). \tag{C.9}$$

To find the coefficients a_n, we multiply both sides of this expression by $\mathrm{P}_m(x)$ and integrate to obtain

$$\int_{-1}^{1} \mathrm{P}_m(x)f(x)\ \mathrm{d}x = \sum_0^\infty a_n \int_{-1}^{1} \mathrm{P}_n(x)\,\mathrm{P}_m(x)\ \mathrm{d}x = a_m\frac{2}{2m+1},$$

so that

$$a_n = \left(n+\frac{1}{2}\right)\int_{-1}^{1} f(x)\,\mathrm{P}_n(x)\ dx. \tag{C.10}$$

C6 A Generating Function for Legendre Polynomials

C6.1 Definition

We consider a function of two variables $G(x,t)$ such that

$$G(x,t) = \sum_{n=0}^\infty \mathrm{P}_n(x)t^n, \tag{C.11}$$

so that the Legendre Polynomials are the coefficients in the Taylor series of $G(x,t)$ about $t=0$. Our first task is to identify what the function $G(x,t)$ actually is.

C6.2 Derivation of the generating function.

We know that, in spherical polar coordinates, the function r^{-1} is harmonic, away from $r=0$, i.e.

$$\nabla^2\frac{1}{r} = \nabla^2\frac{1}{|\underline{\mathrm{x}}|} = 0.$$

It is a harmonic function independent of ϕ. Similarly $1/(|\underline{x}-\underline{x}_0|)$ is harmonic away from $\underline{x}=\underline{x}_0$. If $\underline{x}_0$ is a unit vector in the z-direction, then

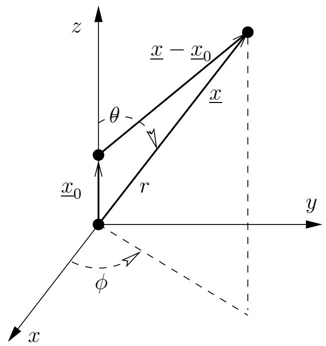
$$\begin{aligned} |\underline{x}-\underline{x}_0|^2 &= (\underline{x}-\underline{x}_0).(\underline{x}-\underline{x}_0) \\ &= \underline{x}.\underline{x} - 2\underline{x}.\underline{x}_0 + \underline{x}_0.\underline{x}_0 \\ &= r^2 - 2r\cos\theta + 1. \end{aligned} \tag{C.12}$$

So the function

$$\frac{1}{\sqrt{r^2-2r\cos\theta+1}}$$

is harmonic, regular at the origin and independent of ϕ. We should therefore be able to write it in the form,

$$\frac{1}{\sqrt{r^2-2r\cos\theta+1}} = \sum_{n=0}^\infty A_n r^n\,\mathrm{P}_n(\cos\theta). \tag{C.13}$$



If we can show that $A_n = 1$, then the replacement of $\cos\theta$ by x and r by t gives the required result. To find A_n, we evaluate the function along the positive z-axis, putting $\cos\theta = 1$, noting that

$$\frac{1}{\sqrt{r^2 - 2r + 1}} = \frac{1}{\sqrt{(r-1)^2}} = \frac{1}{|r-1|} = \frac{1}{1-r} = 1 + r + r^2 + r^3 + \cdots, \quad |r| < 1.$$

So

$$\sum_{n=0}^{\infty} r^n = \sum_{n=0}^{\infty} A_n r^n \, \mathrm{P}_n(1) = \sum_{n=0}^{\infty} A_n r^n,$$

so that $A_n = 1$. Thus, with $x = \cos\theta$ and $t = r$,

$$G(x,t) = \frac{1}{\sqrt{1 - 2xt + t^2}} = \sum_{n=0}^{\infty} t^n \, \mathrm{P}_n(x). \tag{C.14}$$

C6.3 Applications of the Generating Function.

Generating functions can be applied in many ingenious ways, somethimes best left for examination questions. As an example, we can differentiate $G(x,t)$ with respect to t to show that

$$\frac{\partial G}{\partial t} = \frac{x - t}{(1 - 2xt + t^2)^{3/2}} \quad \Longrightarrow \quad (1 - 2xt + t^2)\frac{\partial G}{\partial t} = (x - t)G.$$

Now write $G(x,t)$ as a sum of Legendre polynomials to get

$$(1 - 2xt + t^2) \sum_{n=0}^{\infty} n \, \mathrm{P}_n(x) t^{n-1} = (x - t) \sum_{n=0}^{\infty} \mathrm{P}_n(x) t^n.$$

Now comparing the coefficients of t^0 gives

$$\mathrm{P}_1(x) = x \, \mathrm{P}_0(x),$$

so that, as $\mathrm{P}_0 = 1$, we have $\mathrm{P}_1 = x$, as expected. Comparing the general coefficient of t^n, $n > 0$, gives

$$(n+1)\,\mathrm{P}_{n+1} - 2xn\,\mathrm{P}_n + (n-1)\,\mathrm{P}_{n-1} = x\,\mathrm{P}_n - \mathrm{P}_{n-1},$$

or, rearranging

$$(n+1)\,\mathrm{P}_{n+1} - (2n+1)x\,\mathrm{P}_n + n\,\mathrm{P}_{n-1} = 0, \tag{C.15}$$

a recursion relation for Legendre polynomials.

Differentiating $G(x,t)$ with respect to x, and proceeding in a similat way yields the result

$$\mathrm{P}'_{n+1} - 2x\,\mathrm{P}'_n + \mathrm{P}'_{n-1} = \mathrm{P}_n, \quad n \geq 1. \tag{C.16}$$

Combining (C.15) and (C.16), or obtaining a relationship between G_x and G_t, shows

$$\mathrm{P}'_{n+1} - \mathrm{P}'_{n-1} = (2n+1)\,\mathrm{P}_n\,. \tag{C.17}$$

These need not be learnt.

C6.4 Solution of Laplace's equation.

Remember where we first came across Lagrange polynomials. If $\nabla^2 u = 0$ and u is regular at $\theta = 0, \pi$ in spherical polar coordinates and with $\partial/\partial\phi = 0$ then

$$u(r,\theta) = \sum_{n=0}^{\infty} \left(A_n r^n + \frac{B_n}{r^{n+1}} \right) P_n(\cos\theta). \tag{C.18}$$

C7 Example: Temperatures in a Sphere

The steady temperature distribution $u(\underline{x})$ inside the sphere $r = a$, in spherical polar coordinates, satisfies $\nabla^2 u = 0$. If we heat the surface of the sphere so that $u = f(\theta)$ on $r = a$ for some function $f(\theta)$, what is the temperature distribution within the sphere?

The equation and boundary conditions do not depend on ϕ so we know that u is of the form (C.18). Further more we expect u to be finite at $r = 0$ so that $B_n = 0$. We find the coefficients A_n by evaluating this on $r = a$. We require

$$f(\theta) = \sum_{n=0}^{\infty} A_n a^n \, \mathrm{P}_n(\cos\theta).$$

We can find A_n using the orthogonality of the polynomials (C.7). However in (C.7), the integration is with respect to x and not $\cos\theta$. If $x = \cos\theta$, then $\mathrm{d}x = -\sin\theta \, \mathrm{d}\theta$. The interval $-1 \leq x \leq 1$ is the interval $-\pi \geq \theta \geq 0$. Multiply through by $-\sin\theta\,\mathrm{P}_m(\theta)$ and integrate in θ to obtain

$$\begin{aligned} \int_{\pi}^{0} -\sin\theta f(\theta)\,\mathrm{P}_m(\cos\theta)\,\mathrm{d}\theta &= \sum_{n=0}^{\infty} (a^n A_n) \int_{\pi}^{0} -\sin\theta\,\mathrm{P}_n(\cos\theta)\,\mathrm{P}_m(\cos\theta)\,\mathrm{d}\theta \\ &= \sum_{n=0}^{\infty} (a^n A_n) \int_{-1}^{1} \mathrm{P}_n(x)\,\mathrm{P}_m(x)\,\mathrm{d}x = \frac{2a^m A_m}{2m+1}. \end{aligned} \tag{C.19}$$

So

$$u(r,\theta) = \sum_{n=0}^{\infty} \left(n + \frac{1}{2}\right) \left(\frac{r}{a}\right)^n \mathrm{P}_n(\cos\theta) \int_0^{\pi} f(\nu)\,\mathrm{P}_n(\cos\nu)\sin\nu\,\mathrm{d}\nu. \tag{C.20}$$

Let us heat the northern hemisphere and leave the southern half cold, so that $f(\theta) = 1$ for $0 \leq \theta \leq \pi/2$ and $f(\theta) = 0$ for $\pi/2 < \theta \leq \pi$. Then the integral in (C.20) is

$$\int_0^{\pi/2} \sin\nu\,\mathrm{P}_n(\cos\nu)\,\mathrm{d}\nu = \int_0^1 \mathrm{P}_n(x)\,\mathrm{d}x.$$

An integration of (C.17) gives

$$(2n+1)\int_x^1 \mathrm{P}_n(q)\,\mathrm{d}q = [\mathrm{P}_{n+1} - \mathrm{P}_{n-1}]_x^1 = \mathrm{P}_{n-1}(x) - \mathrm{P}_{n+1}(x), \quad n > 1.$$

We know that $\int_0^1 \mathrm{P}_0(q)\,\mathrm{d}q = 1$ so

$$u(r,\theta) = \frac{1}{2} + \frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n \mathrm{P}_n(\cos\theta)(\mathrm{P}_{n-1}(0) - \mathrm{P}_{n+1}(0)).$$

Note that the temperature at the centre of the hemisphere ($r = 0$) is $1/2$, which might be expected. The Legendre polynomials of odd degree are odd and will be zero at the origin so that the coefficients in the sum will be zero for even values of n. Hence

$$u(r,\theta) = \frac{1}{2} + \frac{1}{2}\left(\frac{r}{a}\right)\sum_{m=0}^{\infty}\left(\frac{r}{a}\right)^{2m} \mathrm{P}_{2m+1}(\cos\theta)(\mathrm{P}_{2m}(0) - \mathrm{P}_{2(m+1)}(0)). \tag{C.21}$$

We need the values at the origin of the polynomials of even degree. Putting $x = 0$ in (C.14) gives

$$\begin{aligned}
\sum_{n=0}^{\infty} t^n\,\mathrm{P}_n(0) &= \frac{1}{\sqrt{1+t^2}} \qquad\qquad \text{(C.22)}\\
&= 1 - \frac{1}{2}t^2 + \frac{1.3}{2.2}\frac{t^4}{2!} - \frac{1.3.5}{2.2.2}\frac{t^6}{3!} + \cdots\\
&= \sum_{m=0}^{\infty}(-1)^m\frac{(2m-1)(2m-3)\cdots 3.1}{2^m m!}t^{2m}\\
&= \sum_{m=0}^{\infty}(-1)^m\frac{(2m)!}{2^{2m}(m!)^2}t^{2m}.
\end{aligned}$$

Therefore

$$\begin{aligned}
\mathrm{P}_{2m}(0) - \mathrm{P}_{2(m+1)}(0) &= (-1)^m\frac{(2m)!}{2^{2m}(m!)^2} - (-1)^{m+1}\frac{(2m+2)!}{2^{2m+2}((m+1)!)^2}\\
&= (-1)^m\frac{(2m)!}{2^{2m}(m!)^2}\left(1 + \frac{2m+1}{2m+2}\right),
\end{aligned}$$

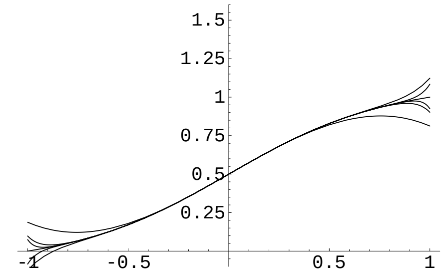
and

$$u(r,\theta) = \frac{1}{2} + \frac{1}{2}\left(\frac{r}{a}\right)\sum_{m=0}^{\infty}\left(\frac{r}{a}\right)^{2m}\mathrm{P}_{2m+1}(\cos\theta)(-1)^m\frac{(2m)!}{2^{2m}(m!)^2}\left(1 + \frac{2m+1}{2m+2}\right).$$

We will evaluate this expression along the axis of the sphere. Here $\cos\theta = \pm 1$, depending if we are in the northern or southern hemisphere. The polynomials in appearing are odd so take the value ± 1 at ± 1. This can be accounted for by allowing r to be negative so that it measures the directed distance from the centre in a northerly direction.

$$u(r) = \frac{1}{2} + \frac{1}{2}\left(\frac{r}{a}\right)\sum_{m=0}^{\infty}\left(\frac{r}{a}\right)^{2m}(-1)^m\frac{(2m)!}{2^{2m}(m!)^2}\left(1 + \frac{2m+1}{2m+2}\right).$$

A graph of the solution obtained by summing this series to 1,4,7,10,13 terms is shown below



We can see that the convergence is not good near the poles. The line that actually attains the values 0 and 1 at the poles is the exact solution

$$u(r) = \frac{1}{2} + \frac{(r/a)^2 + \sqrt{1+(r/a)^2} - 1}{2(r/a)\sqrt{1+(r/a)^2}}.$$

This is attained as follows. Equation (C.21) tells us

$$u(r) = \frac{1}{2} + \frac{1}{2}\left(\frac{r}{a}\right)\sum_{m=0}^{\infty}\left(\frac{r}{a}\right)^{2m}(\mathrm{P}_{2m}(0) - \mathrm{P}_{2(m+1)}(0)).$$

Identifying t with (r/a), and using (C.22),χ recognising that the sum only contains the even powers of t, gives

$$\sum_{m=0}^{\infty}(r/a)^{2m}\,\mathrm{P}_{2m}(0) - \sum_{m=0}^{\infty}(r/a)^{2m-2}\,\mathrm{P}_{2m}(0) = \frac{1}{\sqrt{1+(r/a)^2}} - \frac{1}{(r/a)^2}\frac{1}{\sqrt{1+(r/a)^2}}$$

Changing the index in the second sum, using $\mathrm{P}_0 = 1$, we find

$$\sum_{m=0}^{\infty}(r/a)^{2m}(\mathrm{P}_{2m}(0) - \mathrm{P}_{2m+2}(0)) - \frac{1}{(r/a)^2} = \frac{1}{\sqrt{1+(r/a)^2}} - \frac{1}{(r/a)^2}\frac{1}{\sqrt{1+(r/a)^2}}.$$

And so

$$u(r) = \frac{1}{2} + \frac{1}{2}\frac{r}{a}\left(\frac{1}{\sqrt{1+(r/a)^2}} - \frac{1}{(r/a)^2\sqrt{1+(r/a)^2}} + \frac{1}{(r/a)^2}\right),$$

which simplifies to the result above.

Chapter D

Oscillation of a circular membrane

D1 The Problem and the initial steps in its solution

D1.1 The problem

If we have a circular drum, radius a, and hit it, we will set the drum vibrating. To study this vibration, we need to solve the wave equation

$$c^2\nabla^2\psi = \psi_{tt}, \quad 0 \le r \le a \tag{D.1}$$

with c the speed of wave motion in the drum's material and ψ the displacement of the drum's surface. We have the boundary condition

$$\psi(a,\theta) = 0, \quad 0 \le \theta < 2\pi, \tag{D.2}$$

corresponding to the drum being fixed at its circular edge. We also expect ψ to be finite at $r = 0$, the drum's centre. It we "hit" the drum, so that, at $t = 0$, $\psi = 0$, $\psi_t(r,\theta) = f(r)$, then we must also impose this initial condition.

D1.2 Separating out the time dependence

We look for oscilliatory solutions, writing

$$\psi(r,\theta,t) = u(r,\theta)\exp(i\omega t). \tag{D.3}$$

This is equivalent to looking for solutions of the type $\psi = u(\underline{x})T(t)$ and knowing in advance that the equation for T will have the form $T'' + \omega^2 T = 0$ where ω is related to the separation constant. This has solutions proportional to $\cos\omega t$ and $\sin\omega t$ and we choose to write these in exponential form. The value of ω is the frequency of the disturbance. Doing so leads to

$$c^2\nabla^2 u = -\omega^2 u, \quad \text{or} \quad \nabla^2 u + \lambda^2 u = 0, \tag{D.4}$$

(after dividing through by the exponential factor). Here $\omega = \lambda c$ and we need to solve Helmholtz' equation for the spatial part of the solution. We should expand a little on the shorthand, $\exp(i\omega t)$, that we are using to describe the temporal part of the solution. The solution of the equation for T is $T = A_\omega\cos(\omega t) + B_\omega\sin(\omega t)$. To write this in exponential form we can take the real part of $(A_\omega - iB_\omega)(\cos\omega t + i\sin\omega t)$, i.e. of $(A_\omega - iB_\omega)\exp(i\omega t)$. We therefore have a complex constant multiplying (D.3) which we have omitted. We can write this complex constant in modulus-argument form as $R_\omega\exp(i\epsilon_\omega t)$.

Note that, defining $\omega^2 = \lambda c^2$, we would have ended up with λ rather than λ^2 in (D.4), but that would lead to lots of $\sqrt{}$'s in what follows. Note now that to find the frequency of oscillation we need to solve the eigenvalue problem (D.4), with $u(a,\theta) = 0$ (from (D.2) and u finite at $r = 0$.

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} + \lambda^2 u = 0. \tag{D.5}$$

D1.3 Separating out the θ dependence

We are looking for solitions that are 2π-periodic in θ as we have a circular drum. We know that if we look for solutions of the form $u(r,\theta) = R(r)\Theta(\theta)$ then the θ-dependence will satisfy $\Theta'' + p^2\Theta = 0$, where to further ensure periodicity in θ, $p^2 \ge 0$ and finally to ensure 2π-periodicity $p = n$ for integer n, $n = 0, 1, 2, 3, \ldots$. The case $n = 0$ corresponds to solutions with no θ-dependence. In general the θ-dependence of the solution is like $\sin(n\theta)$ and $\cos(n\theta)$. We choose to write both of these together as $\exp(in\theta)$ (strictly $R_n\exp(in\theta)\exp(i\epsilon_n)$) and look for solutions of the type $u = R(r)\exp(in\theta)$. Subsitution into (D.5), dividing out the exponential factor and multiply by r^2 gives

$$r^2R'' + rR' + (\lambda^2 r^2 - n^2)R = 0, \tag{D.6}$$

and if we write $z = \lambda r$, $R = w(z)$,

$$z^2w'' + zw' + (z^2 - n^2)w = 0. \tag{D.7}$$

We have obtained solutions of this equation for integer n through series and found that it has two independent solutions. We can then write

$$w(z) = A_n\,\mathrm{J}_n(z) + B_n\,\mathrm{Y}_n(z). \tag{D.8}$$

The point $z = 0$ corresponds to the centre of the membrane and we wish our solution to be analytic here. This means that we must set $B_n = 0$ and we consider only solutions finite at $z = 0$, $z = A_n\,\mathrm{J}_n(z)$.

At this stage our solution is of the form

$$\psi(r,\theta,t) = \sum_\lambda \sum_{n=0}^{\infty} A_{n\lambda}\,\mathrm{J}_n(\lambda r)\exp(in\theta)\exp(ic\lambda t) \tag{D.9}$$

where $A_{n\lambda} = R_{n\lambda}\exp(i\epsilon_\lambda)\exp(i\epsilon_n)$ are (complex) constants that we need to find so as to satisfy the initial conditions. In relating this to our definitions above we have used $R_{n\lambda} = R_\omega R_n$. We return to this solution later, but for the present we look more closely at the properties of the solutions of Bessel's equation so as to firstly enable us to fix possible values of λ and secondly to express the initial conditions as generalised Fourier Series.

D1.4 On Bessel Functions

D1.4a Expression as a series

For general p we have the series solution

$$J_p(x) = \sum_{j=0}^{\infty} \frac{(-1)^j}{\Gamma(j+1)\Gamma(j+p+1)}\left(\frac{x}{2}\right)^{2j+p} \tag{D.10}$$

with $\Gamma(z)$ the Gamma function, extending the factorial function to non-integer argument and with $n! = \Gamma(n+1)$. The second independent solution is $J_{-p}(x)$ if p *is not an integer.* However if p is an integer then as $J_{-n} = (-1)^n J_n$, these solutions are linearly dependent. This can be seen as follows. We have the series

$$\mathrm{J}_{\pm\nu} = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!(r \pm \nu)!} \left(\frac{z}{2}\right)^{2r \pm \nu}, \nu = 0, 1, 2, \cdots .$$

If we look at the solution $\mathrm{J}_{-\nu}$, and recall that $x! = \Gamma(1+x)$ has singularities at $x = -1, -2, -3, \cdots$ then we realise that the first few terms in the sum become zero as $\nu \to n$ an integer. These terms correspond to $r - n = -1, -2, -3, \cdots, -n$, corresponding to $r = n-1, n-2, \cdots, 0$. Thus, if ν is an integer we need start the series at $r = n$ for J_{-n}. This gives

$$\mathrm{J}_{-n}(z) = \sum_{r=n}^{\infty} \frac{(-1)^r}{r!(r-n)!} \left(\frac{z}{2}\right)^{2r-n} = \sum_{r=0}^{\infty} \frac{(-1)^r(-1)^n}{(r+n)!(r)!} \left(\frac{z}{2}\right)^{2r+n} = (-1)^n \, \mathrm{J}_n(z),$$

using $r^{\text{old}} = r^{\text{new}} + n$. To cover this case also a second linearly independent solution $Y_p(x)$ is taken. This is defined as

$$Y_p(x) = \frac{J_p \cos p\pi - J_{-p}(x)}{\sin p\pi} \tag{D.11}$$

and the limit $p \to n$ considered if $p = n$. (Note Y_p is often written N_p.) For small x

$$J_p(x) \approx \frac{1}{\Gamma(p+1)} \left(\frac{x}{2}\right)^p \tag{D.12}$$

$$Y_0(x) \approx (2/\pi)\,(\ln(x/2) + \gamma + \ldots) \tag{D.13}$$

$$Y_p(x) \approx -\frac{\Gamma(p)}{\pi} \left(\frac{2}{x}\right)^p \tag{D.14}$$

D1.4b Behaviour for large x.

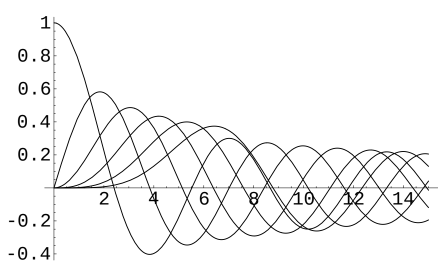
For large x all these solutions behave like a damped sinusoidal function with $y(x) \approx A\sin(x+\epsilon)/\sqrt{x}$. This is easy to demonstrate, writing $w(x) = f(x)y(x)$ and substituting into (D.7)

$$x^2(f''y + 2f'y' + fy'') + x(f'y + fy') + (x^2 - n^2)fy = 0$$

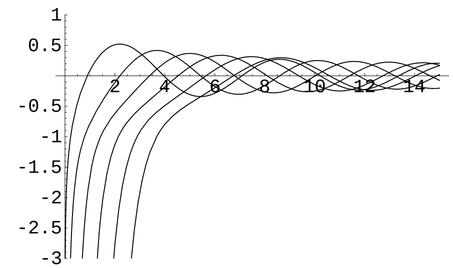
The coeficient of y' can be made zero if $2x^2 f' + xf = 0$ giving $f \propto x^{-1/2}$. Choosing $f = x^{-1/2}$ and dividing by $x^{3/2}$ gives

$$y'' + \left(1 + \frac{1/4 - n^2}{x^2}\right) y = 0.$$

For large x, this is well approximated by $y'' + y = 0$ so that $y = A\sin(x+\epsilon)$ and the result.




```
Table[Plot[BesselJ[i, x], {x, 0, 15}, PlotPoints -> 50], {i, 0, 5}];Show[%]
```




```
Table[Plot[BesselY[i, x], {x, 0, 15}, PlotPoints -> 50], {i, 0, 5}];
Show[%,PlotRange -> {-3, 1}]
```

All the solutions have an *infinite number of zeros.* The zeros of J_n are denoted j_{nm} so that $J_n(j_{nm}) = 0$ and $0 < j_{n1} < j_{n2} < j_{n3} < \ldots$.

	j_{n1}	j_{n2}	j_{n3}	j_{n4}	j_{n5}	$\cdots$
$n=0$	2.4048	5.5201	8.6537	11.7915	14.9309	$\cdots$
$n=1$	3.8317	7.0156	10.173	13.323	16.471	$\cdots$
$n=2$	5.1356	8.4172	11.6198	14.796	17.960	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	

D1.5 Determination of λ

We have the boundary condtion $\psi(a,\theta) = 0$, i.e. $w(\lambda a) = 0$. Thus λ is determined so that $\lambda a = j_{nm}$ or $\lambda = j_{nm}/a$, $m = 1, 2, 3, \cdots$. Thus the possible frequencies of the drum's vibration are determined as the doubly infinite family

$$\omega = \omega_{nm} = c j_{nm}/a \tag{D.15}$$

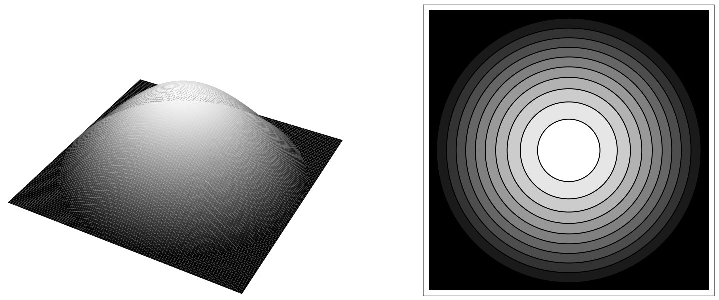
For each frequency the vibration is described by

$$\mathrm{J}_n(j_{nm} r/a) \exp(icj_{nm}t/a) \exp(in\theta) \tag{D.16}$$

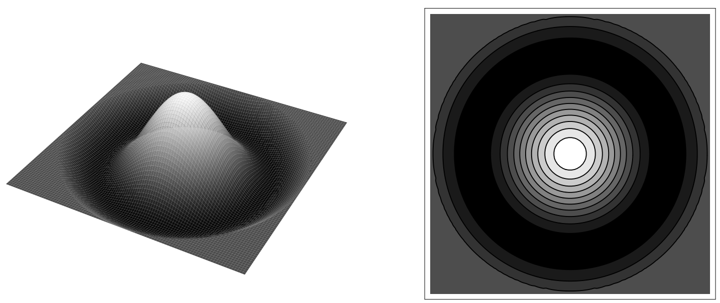
The general solution is an arbitray linear combination of all these modes so that (D.9) becomes the double sum

$$\psi(r,\theta,t) = \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} A_{nm} \, \mathrm{J}_n(j_{nm} r/a) \exp(in\theta) \exp(icj_{nm}t/a). \tag{D.17}$$

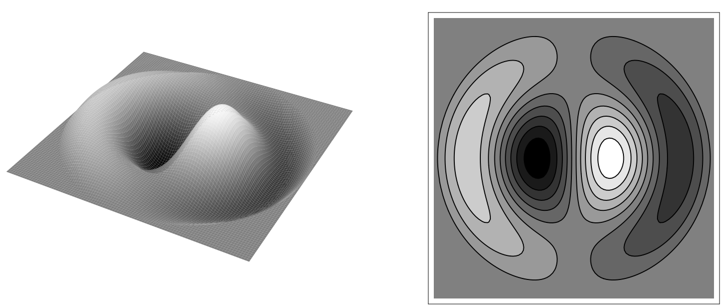
Here now $A_{nm} = R_{nm} \exp(i\epsilon_n) \exp(i\epsilon_m)$. The constants R_{nm} can be related to the amplitude of a particular mode, ϵ_n to its orientation realtive to the line $\theta = 0$ and ϵ_m to its phase (where in the sinusoidal temporal cycle it started).



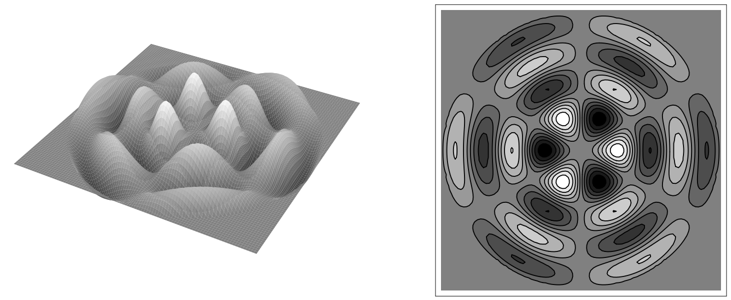
$n = 0$, $m = 1$



$n = 0$, $m = 2$



$n = 1$, $m = 2$



$n = 3$, $m = 4$

```
<< NumericalMath`BesselZeros`;n = 1; m = 2;jnm = BesselJZeros[n, m][[m]];
radial[r_] = BesselJ[n, jnm r];azimuth[theta_] = Re[Exp[I n theta]];
wrap[f_, r_, theta_] = If[r < 1, f[r, theta], 0];polarr[x_, y_] = Sqrt[x^2 + y^2];
polartheta[x_, y_] = If[x > 0, ArcTan[y/x], If[y > 0, Pi/2 + ArcTan[-x/y],
-Pi/2 - ArcTan[x/y]]];mode[r_, theta_] = radial[r]azimuth[theta];
surf = Plot3D[wrap[mode, polarr[x, y], polartheta[x,y]], {x, -1, 1},{y, -1, 1},
PlotRange -> All, PlotPoints -> {100,100}, Lighting -> False, Mesh ->False,
Axes -> False, Boxed -> False];cont = ContourPlot[wrap[mode, polarr[x, y],
polartheta[x, y]], {x, -1,1}, {y, -1, 1}, PlotRange -> All, PlotPoints-> {100, 100},
FrameTicks->None];Show[GraphicsArray[{surf, cont}]]
```

D1.6 Incorporating the Initial Conditions

Our initital conditions are that, at $t = 0$, $\psi = 0$ and $\psi_t = f(r)$. Putting $t = 0$ in (D.17) gives

$$0 = \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} A_{nm} \, \mathrm{J}_n(j_{nm} r/a) \exp(in\theta).$$

Differentiating and putting $t = 0$ in (D.17)

$$f(r) = \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} (icj_{nm}/a) A_{nm} \, \mathrm{J}_n(j_{nm} r/a) \exp(in\theta)$$

where A_{nm} are complex constants (as above) and it is understood that we take the real part of the sum. The right hand side has no θ-dependence and we can deduce that we need only the component $n = 0$ from the first sum. Thus we must find $A_{0m} = A_m = R_m \exp(i\epsilon_m)$, say, such that, taking real parts,

$$0 = \sum_{m=1}^{\infty} R_m \exp(i\epsilon_m) \, \mathrm{J}_n(j_{0m} r/a). \tag{D.18}$$

$$f(r) = \sum_{m=1}^{\infty} (icj_{0m}/a) R_m \exp(i\epsilon_m) \, \mathrm{J}_n(j_{0m} r/a). \tag{D.19}$$

This can be achieved by setting $\exp(i\epsilon_m) = -i$, remembering that R_m is real. The choice $-i$ is driven by a desire for neatness in (D.19).

Note that we have been a little overgeneral in our presentation here. Right at the start we could have described the time-dependence by the solution $\sin(\omega t)$. This is zero but has a non-zero time derivative at $t = 0$ and is obviously that which is required for our particular initial conditions. This corresponds to our current choice of a purely imaginary value for $\exp(i\epsilon_m)$. Similarly we could have realised that, as the initial and boundary conditions had no θ-dependence, neither the final solution and included only the $n = 0$ mode from the start. If there was some θ-dependence in the initial condition, then we would deal with each Fourier component (in θ) separately. We are looking to represent $f(r)$ as a generalised Fourier Series in r, otherwise known as a Fourier-Bessel Series. This is possible as the differential equation satisfied by the Bessel functions, together with the boundary conditions - finiteness at $r = 0$ and 0 at $r = a$ are a Sturm-Liouvuille problem.

D1.7 Orthogonality of Bessel Functions

The set of functions $\mathrm{J}_n(\lambda r)$ in $0 \le r \le a)$ with $\lambda = j_{0m}/a$ are eigenfunctions of the Sturm-Liouiville problem

$$r^2R'' + rR' + (\lambda^2 r^2 - n^2)R = 0, \quad R(0) \quad \text{finite}, \quad R(a) = 0 \tag{D.20}$$

We can rewite this as

$$[rR']' + \left(\lambda^2 r - \frac{n^2}{r}\right) R = 0,$$

with solution $R = \mathrm{J}_n(\lambda r)$. Let $\lambda = \lambda_i$ be such that $\mathrm{J}_n(\lambda_i a) = 0$ and λ be a different value. We have

$$[r\,\mathrm{J}_n(\lambda_i r)']\,\prime \quad + \quad \left(\lambda_i^2 r - \frac{n^2}{r}\right) \mathrm{J}_n(\lambda_i r) = 0, \tag{D.21}$$

$$[r\,\mathrm{J}_n(\lambda r)']\,\prime \quad + \quad \left(\lambda^2 r - \frac{n^2}{r}\right) \mathrm{J}_n(\lambda r) = 0. \tag{D.22}$$

If we multiply (D.21) by $\mathrm{J}_n(\lambda r)$, (D.22) by $\mathrm{J}_n(\lambda_i r)$, subtract and integrate we get

$$\begin{aligned}\int_0^a \mathrm{J}_n(\lambda r)\,[r\,\mathrm{J}_n(\lambda_i r)']' \quad - \quad \mathrm{J}_n(\lambda_i r)\,[r\,\mathrm{J}_n(\lambda r)']'\ \mathrm{d}r + (\lambda_i^2 - \lambda^2)\int_0^a r\,\mathrm{J}_n(\lambda_i r)\,\mathrm{J}_n(\lambda r)\ \mathrm{d}r \\ - \quad n^2 \int_0^a \underbrace{(\mathrm{J}_n(\lambda_i r)\,\mathrm{J}_n(\lambda r) - \mathrm{J}_n(\lambda r)\,\mathrm{J}_n(\lambda_i r))}_{=0}\ \frac{\mathrm{d}r}{r} = 0.\end{aligned} \tag{D.23}$$

Use integration by parts on the first integral to get

$$\begin{aligned}[\mathrm{J}_n(\lambda r)\,\{r\,\mathrm{J}_n(\lambda_i r)'\} - \mathrm{J}_n(\lambda_i r)\,\{r\,\mathrm{J}_n(\lambda r)'\}] \quad - \int_0^a \underbrace{\mathrm{J}_n(\lambda r)' r\,\mathrm{J}_n(\lambda_i r)' - \mathrm{J}_n(\lambda_i r)' r\,\mathrm{J}_n(\lambda r)'}_{=0}\ \mathrm{d}r \\ + (\lambda_i^2 - \lambda^2)\int_0^a r\,\mathrm{J}_n(\lambda_i r)\,\mathrm{J}_n(\lambda r)\ \mathrm{d}r = 0.\end{aligned}$$

Now $\mathrm{J}_n(\lambda r)' = \lambda\,\mathrm{J}_n'(\lambda r)$ so, using the fact that $J_n(\lambda_i a) = 0$, we have the result

$$\int_0^a r\,\mathrm{J}_n(\lambda_i r)\,\mathrm{J}_n(\lambda r)\ \mathrm{d}r = a\lambda_i\,\mathrm{J}_n'(\lambda_i a)\frac{\mathrm{J}_n(\lambda a)}{\lambda^2 - \lambda_i^2}. \tag{D.24}$$

Thus if $\lambda \ne \lambda_i$ and $\mathrm{J}(\lambda a) = 0$, i.e. λ is another, distinct, eigenvalue, then

$$\int_0^a r\,\mathrm{J}_n(\lambda_i r)\,\mathrm{J}_n(\lambda r)\ \mathrm{d}r = 0. \tag{D.25}$$

To see what happens if $\lambda = \lambda_i$, let $\lambda \to \lambda_i$ and use l'Hopital's rule.

$$\begin{aligned}\int_0^a r\,\mathrm{J}_n(\lambda_i r)^2\ \mathrm{d}r &= a\lambda_i\,\mathrm{J}_n'(\lambda_i a)\frac{\frac{\partial}{\partial\lambda}\,\mathrm{J}_n(\lambda a)\big|_{\lambda=\lambda_i}}{\frac{\partial}{\partial\lambda}(\lambda^2 - \lambda_i^2)\big|_{\lambda=\lambda_i}} \\ &= \frac{a^2}{2}\left[\mathrm{J}_n'(\lambda_i a)\right]^2.\end{aligned} \tag{D.26}$$

We can take λ and λ_i as different roots of J_0, j_{0m}. We shall see in the section on the generating function that follows that

$$\mathrm{J}_{n\pm1}(x) = \frac{n}{x}\,\mathrm{J}_n(x) \mp \mathrm{J}_n'(x), \quad \Longrightarrow (\mathrm{J}_0')^2 = (\mathrm{J}_1)^2,$$

and we can easily evaluate the values of J_1 at the zeros of J_0.

D1.8 The solution

We can now consider (D.19), multiply by $r\,\mathrm{J}_0(rj_{0n}/a)$ and integrate between 0 and a. Only one element of the sum survives due to the orthogonality and we have

$$\int_0^a r\,\mathrm{J}(rj_{0n}/a)f(r)\ \mathrm{d}r = R_n(cj_{0n}/a)(a^2/2)[\mathrm{J}_1(j_{0n})]^2,$$

giving R_n We make the substitution for $A_{0m} = R_m \exp(i\epsilon_m) = -iR_m$ in (D.17) and take the real part of the result to yeild

$$\psi(r,\theta,t) = \sum_{m=1}^{\infty} \frac{2\int_0^a r\,\mathrm{J}(rj_{0m}/a)f(r)\ \mathrm{d}r}{cj_{0m}a[\mathrm{J}_1(j_{0m})]^2}\,\mathrm{J}_0(rj_{0m}/a)\sin(cj_{0m}t/a). \tag{D.27}$$

The animation at `http://www.ucl.ac.uk/~ucahdrb/MATHM242/` illustrates this solution and used the following commands

```
<< NumericalMath`BesselZeros`;f[x_] = Sin[2Pi x]; stop = 10; c = 1;j0m = BesselJZeros[0,
stop]; coef = Map[2NIntegrate[x BesselJ[0, # x] f[x], {x, 0, 1}]/(c# BesselJ[1, #]^2) &,
j0m]; soln[r_, t_] := Tr[coef*Map[BesselJ[0, r#]&, j0m]*Map[Sin[c # t] &, j0m]];
res[t_] := Module[{},wrap[f_, r_]=If[r < 1, f[r, t], 0]; polarr[x_, y_] = Sqrt[x^2 + y^2];
surf = Plot3D[wrap[soln, polarr[x, y]], {x, -1, 1}, {y, -1, 1}, PlotRange -> {-.3, .3},
PlotPoints -> {20, 20},Lighting -> False, Mesh -> False, Axes -> False, Boxed -> False]];
Export["res.gif", Table[res[i], {i, 0, 6, .01}]]
```

D1.8a Generating Function

There is a generating function for the Bessel functions J_n. It turns out that

$$\sum_{n=-\infty}^{n=\infty} J_n(x)t^n = \exp\left(\frac{x}{2}\left(t - \frac{1}{t}\right)\right). \tag{D.28}$$

This can be derived by considering solutions of Helmholtz' equation for $G(\underline{x})$ in Cartesian and in polar coordinates.

$$\nabla^2 G + G = 0, \quad G_{xx} + G_{yy} + G = 0, \quad r^2 G_{rr} + rG_r + G_{\theta\theta} + r^2 G = 0.$$

One solution is $G = \exp(iy)$, corresponding to a plane wave. In polar coordinates, this is $G = \exp(ir\sin\theta)$. We have seen that the solution in polar coordinates can be found by separating out the angular dependence $\exp(in\theta)$ leaving the radial dependence satisfying Bessel's equation. The plane wave is regular at the origin so we do not need the Y_n solutions. We know from section A8 that we should be able to write G as follows

$$G = \exp(ir\sin\theta) = \sum_{n=-\infty}^{\infty} A_n \,\mathrm{J}_n(r)\exp(in\theta) \tag{D.29}$$

for some constants A_n. If we write $t = \exp(i\theta)$ so that $\sin\theta = (t - t^{-1})/2$, and replace r by x then we have

$$\exp\left(\frac{x}{2}\left(t - \frac{1}{t}\right)\right) = \sum_{n=-\infty}^{n=\infty} A_n J_n(x) t^n. \tag{D.30}$$

To show that the A_n are all equal to one we use our knowledge of the expansions of $J_n(x)$ for small x, obtained from the series solutions. From (D.12)

$$\mathrm{J}_n^n(0) = 2^{-n}.$$

We start by isolating a particular J_m from the sum (D.29) by effectively using the orthogonality properties of $\cos(n\theta)$ and $\sin(n\theta)$ which lie behind the idea of Fourier Series. We multiply (D.29) by $\exp(-im\theta)$ and integrate

$$\int_0^{2\pi} \exp(ir\sin\theta - im\theta)\mathrm{d}\theta = \sum_{n=-\infty}^{\infty} A_n \,\mathrm{J}_n(r) \int_0^{2\pi} \exp(in\theta - im\theta)\mathrm{d}\theta = 2\pi A_m \,\mathrm{J}_m(r)$$

Now differentiate m times with respect to r and put $r = 0$ to find

$$\int_0^{2\pi} i^m(\sin^m\theta)\exp(ir\sin\theta)\mathrm{d}\theta = 2\pi A_m \,\mathrm{J}_m^m(r),$$

$$\implies \int_0^{2\pi} i^m \sin^m\theta \mathrm{d}\theta = 2\pi A_m/2^m.$$

Using the exponential form for $\sin\theta$ gives

$$A_m = \frac{2^m}{2\pi}\int_0^{2\pi}\frac{1}{2^m}(\exp i\theta - \exp -i\theta)^m \exp(-im\theta)\mathrm{d}\theta = \frac{1}{2\pi}\int_0^{2\pi}(1-\exp(-2im\theta))^m\mathrm{d}\theta = \frac{1}{2\pi}2\pi = 1,$$

as the only non-zero contribution to the integral comes from integrating the first term in the binomial expansion of the integrand.

D1.8b Use of the Generating Function

1. We see that

$$G_x = \frac{1}{2}\left(t - \frac{1}{t}\right) G$$

so that

$$\sum_{n=-\infty}^{\infty} t^n \,\mathrm{J}_n'(x) = \frac{1}{2}\sum_{n=-\infty}^{\infty} t^{n+1}\,\mathrm{J}_n(x) - \frac{1}{2}\sum_{n=-\infty}^{\infty} t^{n-1}\,\mathrm{J}_n(x),$$

and comparing coefficients of powers of t,

$$\mathrm{J}_n'(x) = \frac{1}{2}\left(\mathrm{J}_{n-1}(x) - \mathrm{J}_{n+1}(x)\right). \tag{D.31}$$

2. If we make the replacement $t \to -1/t$, we see that

$$G(x, -1/t) = \exp\left(\frac{1}{2}\left(-\frac{1}{t} + t\right)\right) = \exp\left(\frac{1}{2}\left(t - \frac{1}{t}\right)\right) = G(x,t),$$

so that

$$\sum_{n=-\infty}^{\infty} \mathrm{J}_n(x)\frac{(-1)^n}{t^n} = \sum_{n=-\infty}^{\infty} t^n \,\mathrm{J}_n(x).$$

However the left hand side is also $\sum_{n=-\infty}^{\infty} t^n \,\mathrm{J}_{-n}(x)(-1)^n$, using $-n$, rather than n to take us through the sum. Thus

$$\sum_{n=-\infty}^{\infty} t^n \,\mathrm{J}_{-n}(x)\frac{(-1)^n}{=}\sum_{n=-\infty}^{\infty} t^n \,\mathrm{J}_n(x),$$

and comparing coefficients of t^n,

$$\mathrm{J}_n(x) = (-1)^n \,\mathrm{J}_{-n}(x), \tag{D.32}$$

so that

$$\mathrm{J}_{-1}(x) = -\,\mathrm{J}_1(x), \quad \mathrm{J}_{-2}(x) = \mathrm{J}_2(x).$$

Thus from (D.31), with $n = 0$,

$$\mathrm{J}_0'(x) = -\,\mathrm{J}_1(x). \tag{D.33}$$

3. If we choose to differentiate G with respect to t, then we can show

$$\frac{2n}{x}\,\mathrm{J}_n(x) = \mathrm{J}_{n-1}(x) + \mathrm{J}_{n+1}(x). \tag{D.34}$$