PROBLEM OF THE WEEK

Solution of Problem No. 10 (Fall 2013 Series)

Problem:

Show that $\tan^{-1}(k) = \sum_{n=0}^{k-1} \tan^{-1}\left(\frac{1}{n^2+n+1}\right)$ **if** $k \geq 1$**, and deduce that**

$$\sum_{n=0}^{\infty} \tan^{-1}\left(\frac{1}{n^2+n+1}\right) = \pi/2.$$

Solution: (by Prathem D. Ghael, Freshman, Engineering, Purdue University)

We use mathematical induction. Clearly the first formula above holds for $k = 1$. We assume that the formula is true for any $k > 1$. So

$$\tan^{-1}(k) = \sum_{n=0}^{k-1} \tan^{-1}\left(\frac{1}{n^2+n+1}\right) \qquad \text{is true.} \tag{1}$$

We now prove that the formula is true for $k+1$.

So LHS

$\tan^{-1}(k+1)$

RHS

$$\begin{aligned}\sum_{n=0}^{(k+1)-1} \tan^{-1}\left(\frac{1}{n^2+n+1}\right) &= \sum_{n=0}^{k} \tan^{-1}\left(\frac{1}{n^2+n+1}\right) \\ &= \sum_{n=0}^{k-1} \tan^{-1}\left(\frac{1}{n^2+n+1}\right) + \tan^{-1}\left(\frac{1}{k^2+k+1}\right).\end{aligned}$$

By using equation (1), we have

$$= \tan^{-1}(k) + \tan^{-1}\left(\frac{1}{k^2+k+1}\right).$$

Using identity $\rightarrow$ $\tan^{-1}(A) + \tan^{-1}(B) = \tan^{-1}\left(\frac{A+B}{1-AB}\right)$. Thus, we have

$$\tan^{-1}\left(\frac{k+\frac{1}{k^2+k+1}}{1-\frac{k}{k^2+k+1}}\right) = \tan^{-1}\left(\frac{k^3+k^2+k+1}{k^2+1}\right).$$

Using long division, we have $\tan^{-1}(k+1)$. Thus the formula holds for all k.

DEDUCTION

We know that

$\sum_{n=0}^{k-1} \tan^{-1}\left(\frac{1}{n^2+n+1}\right) = \tan^{-1}(k)$ is true. So

$$\begin{aligned}\sum_{n=0}^{\infty} \tan^{-1}\left(\frac{1}{n^2+n+1}\right) &= \lim_{k\to\infty} \tan^{-1}(k) \\ &= \frac{\pi}{2}.\end{aligned}$$

The problem was also solved by:

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