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Algebraic and Geometric Preliminaries

The mathematician Euler once said, "God made integers, all else is the work of man." In this chapter, we have advanced in the evolutionary process to the real number system. We partially characterize the real numbers and then, alas, find an imperfection. The quadratic equation $x^2 + 1 = 0$ has no solution.

A new day arrives, the complex number system is born. We view a complex number in several ways: as an element in a field, as a point in the plane, and as a two-dimensional vector. Each way is useful and in each way we see an unmistakable resemblance of the complex number system to its parent, the real number system. The child seems superior to its parent in every way except one—it has no order. This sobering realization creates a new respect for the almost discarded parent.

The moral of this chapter is clear. As long as the child follows certain guidelines set down by its parent, it can move in new directions and teach us many things that the parent never knew.

1.1 The Complex Field

We begin our study by giving a very brief motivation for the origin of complex numbers. If all we knew were positive integers, then we could not solve the equation $x + 2 = 1$. The introduction of negative integers enables us to obtain a solution. However, knowledge of every integer is not sufficient for solving the equation $2x - 1 = 2$. A solution to this equation requires the study of *rational numbers*.

While all linear equations with integers coefficients have rational solutions, there are some quadratics that do not. For instance, *irrational numbers* are needed to solve $x^2 - 2 = 0$. Going one step further, we can find quadratic equations that have no real (rational or irrational) solutions. The equation $x^2 + 1 = 0$ has no real solutions because the square of any real number is nonnegative. In order to solve this equation, we must "invent" a number whose square is -1. This number, which we shall denote by $i = \sqrt{-1}$, is called an *imaginary unit*.

Our sense of logic rebels against just "making up" a number that solves a particular equation. In order to place this whole discussion in a more rigorous setting, we will define operations involving combinations of real numbers and imaginary units. These operations will be shown to conform, as much as possible, to the usual rules for the addition and multiplication of real numbers. We may express any ordered pair of real numbers (a, b) as the "complex number"

$$a + bi \quad \text{or} \quad a + ib. \tag{1.1}$$

The set of complex numbers is thus defined as the set of all ordered pairs of real numbers. The notion of equality and the operations of addition and multiplication are defined as follows:[1]

$$\begin{aligned}(a_1, b_1) = (a_2, b_2) &\Longleftrightarrow a_1 = a_2,\ b_1 = b_2,\\ (a_1, b_1) + (a_2, b_2) &= (a_1 + a_2, b_1 + b_2),\\ (a_1, b_1)(a_2, b_2) &= (a_1a_2 - b_1b_2, a_1b_2 + a_2b_1).\end{aligned}$$

The definition for the multiplication is more natural than it appears to be, for if we denote the complex numbers of the form (1.1), multiply as we would real numbers, and use the relation $i^2 = -1$, we obtain

$$(a_1 + ib_1)(a_2 + ib_2) = a_1a_2 - b_1b_2 + i(a_1b_2 + a_2b_1).$$

Several observations should be made at this point. First, note that the formal operations for addition and multiplication of complex numbers do not depend on an imaginary number i. For instance, the relation $i^2 = -1$ can be expressed as $(0, 1)(0, 1) = (-1, 0)$. The symbol i has been introduced purely as a matter of notational convenience. Also, note that the order pair $(a, 0)$ represents the real number a, and that the relations

$$(a, 0) + (b, 0) = (a + b, 0) \quad \text{and} \quad (a, 0)(b, 0) = (ab, 0)$$

are, respectively, addition and multiplication of real numbers. Some of the essential properties of real numbers are as follows: Both the *sum* and *product* of real numbers are real numbers, and the *order* in which either operation is performed may be *reversed*. That is, for real numbers a and b, we have the commutative laws

$$a + b = b + a \quad \text{and} \quad a \cdot b = b \cdot a. \tag{1.2}$$

The associative laws

$$a + (b + c) = (a + b) + c \quad \text{and} \quad a \cdot (b \cdot c) = (a \cdot b) \cdot c, \tag{1.3}$$

[1] The symbol $\Longleftrightarrow$ stands for "if and only if" or "equivalent to."

and the distributive law

$$a \cdot (b + c) = a \cdot b + a \cdot c \tag{1.4}$$

also holds for all real numbers a, b, and c. The numbers 0 and 1 are, respectively, the *additive* and *multiplicative identities*. The *additive inverse* of a is $-a$, and the *multiplicative inverse* of $a\,(\neq 0)$ is the real number $a^{-1} = 1/a$. Stated more concisely, the real numbers form a *field* under the operations of addition and multiplication.

Of course, the real numbers are not the only system that forms a field. The rational numbers are easily seen to satisfy the above conditions for a field. What is important in this chapter is that the complex numbers also form a field. The additive identity is $(0, 0)$, and the additive inverse of (a, b) is $(-a, -b)$. The multiplicative inverse of $(a, b) \neq (0, 0)$ is

$$\left(\frac{a}{a^2 + b^2}, -\frac{b}{a^2 + b^2}\right).$$

We leave the confirmation that the complex numbers satisfy all the axioms for a field as an exercise for the reader.

The discerning math student should not be satisfied with the mere verification of a proof. He/she should also have a "feeling" as to why the proof works. Did the reader ask why the multiplicative inverse of (a, b) might be expected to be

$$\left(\frac{a}{a^2 + b^2}, -\frac{b}{a^2 + b^2}\right)?$$

Let us go through a possible line of reasoning. If we write the inverse of $(a, b) = a + bi$ as

$$(a + ib)^{-1} = \frac{1}{a + ib},$$

then we want to find a complex number $c + di$ such that

$$\frac{1}{a + ib} = c + id.$$

By cross multiplying, we obtain $ac + i^2 bd + i(ad + bc) = 1$, or

$$\begin{cases} ac - bd = 1, \\ ad + bc = 0. \end{cases}$$

The solution to these simultaneous equation is

$$c = \frac{a}{a^2 + b^2}, \quad d = -\frac{b}{a^2 + b^2}.$$

Can the reader think of other reasons to suspect that

$$(a,b)^{-1} = \left(\frac{a}{a^2+b^2}, -\frac{b}{a^2+b^2}\right)?$$

Let $z = (x, y)$ be a complex number. Then x and y are called the *real part* of z, $\operatorname{Re} z$, and the *imaginary part* of z, $\operatorname{Im} z$, respectively. Denote the set of real numbers by $\mathbb{R}$ and the set of complex numbers by $\mathbb{C}$. There is a one-to-one correspondence between $\mathbb{R}$ and a subset of $\mathbb{C}$, represented by $x \leftrightarrow (x, 0)$ for $x \in \mathbb{R}$, which preserves the operations of addition and multiplication. Hence we will use the real number x and the ordered pair $(x, 0)$ interchangeably. We will also denote the ordered pair $(0, 1)$ by i. Because a complex number is an ordered pair of real numbers, we use the terms $\mathbb{C} = \mathbb{R}^2$ or $\mathbb{C} = \mathbb{R} \times \mathbb{R}$ interchangeably. Thus $\mathbb{R} \times 0$ is a subset of $\mathbb{C}$ consisting of the real numbers.

As noted earlier, an advantage of the field $\mathbb{C}$ is that it contains a root of $z^2 + 1 = 0$. In Chapter 8 we will show that any polynomial equation $a_0 + a_1 z + \cdots + a_n z^n = 0$ has a solution in $\mathbb{C}$. But this extension from $\mathbb{R}$ to $\mathbb{C}$ is not without drawbacks. There is an important property of the real field that the complex field lacks. If $a \in \mathbb{R}$, then exactly *one* of the following is true:

$$a = 0, \quad a > 0, \quad -a > 0 \quad \text{(trichotomy)}.$$

Furthermore, the sum and the product of two positive real numbers is positive (closure).

A field with an order relation $<$ that satisfies the trichotomy law and these two additional conditions is said to be *ordered*. In an ordered field, like the real or rational numbers, we are furnished with a natural way to compare any two elements a and b. Either a is less than b $(a < b)$, or a is equal to b $(a = b)$, or a is greater than b $(a > b)$. Unfortunately, no such relation can be imposed on the complex numbers, for suppose the complex numbers are ordered; then either i or $-i$ is positive. According to the closure rule, $i^2 = (-i)^2 = -1$ is also positive. But 1 must be negative if -1 is positive. However, this violates the closure rule because $(-1)^2 = 1$.

To sum up, there is a complex field that contains a real field that contains a rational field. There are advantages and disadvantages to studying each field. It is not our purpose here to state properties that uniquely determine each field, although this most certainly can be done.

Questions 1.1.

1. Can a field be finite?
2. Can an ordered field be finite?
3. Are there fields that properly contain the rationals and are properly contained in the reals?
4. When are two complex numbers z_1 and z_2 equal?
5. What complex numbers may be added to or multiplied by the complex number $a + ib$ to obtain a real number?
6. How can we separate the quotient of two complex numbers into its real and imaginary parts?

7. What can we say about the real part of the sum of the two complex numbers? What about the product?
8. What kind of implications are there in defining a complex number as an ordered pair?
9. If a polynomial of degree n has at least one solution, can we say more?
10. If we try to define an ordering of the complex numbers by saying that $(a, b) > (c, d)$ if $a > b$ and $c > d$, what order properties are violated?
11. Can any ordered field have a solution to $x^2 + 1 = 0$?

Exercises 1.2.

1. Show that the set of real numbers of the form $a + b\sqrt{2}$, where a and b are rational, is an ordered field.
2. If a and b are elements in a field, show that $ab = 0$ if and only if either $a = 0$ or $b = 0$.
3. Suppose a and b are elements in an ordered field, with $a < b$. Show that there are infinitely many elements between a and b.
4. Find the values of

   (a) $(-2, 3)(4, -1)$ (b) $(1 + 2i)\{3(2 + i) - 2(3 + 6i)\}$
   (c) $(1 + i)^3$ (d) $(1 + i)^4$
   (e) $(1 + i)^n - (1 - i)^n$.
5. Express the following in the form $x + iy$:

   (a) $(1 + i)^{-5}$ (b) $(3 - 2i)/(1 - i)$
   (c) $e^{i\pi/2} + \sqrt{2}e^{i\pi/4}$ (d) $(1 + i)e^{i\pi/6}$
   (e) $\dfrac{a + ib}{a - ib} - \dfrac{a - ib}{a + ib}$ (f) $\dfrac{3 + 5i}{7 + i} + \dfrac{1 + i}{4 + 3i}$
   (g) $(2 + i)^2 + (2 - i)^2$ (h) $\dfrac{(4 + 3i)\sqrt{3 + 4i}}{3 + i}$
   (i) $\dfrac{(ai^{40} - i^{17})}{-1 + i}$, ($a$–real) (j) $(-1 + i\sqrt{3})^{60}$
   (k) $\dfrac{\sqrt{1 + a^2} + ia}{a - i\sqrt{1 + a^2}}$, ($a$–real) (l) $\dfrac{(\sqrt{3} - i)^2(1 + i)^5}{(\sqrt{3} + i)^4}$.
6. Show that
$$\left(\frac{-1 \pm \sqrt{3}}{2}\right)^3 = 1 \quad \text{and} \quad \left(\frac{\pm 1 \pm i\sqrt{3}}{2}\right)^6 = 1$$
   for all combinations of signs.
7. For any integers k and n, show that $i^n = i^{n+4k}$. How many distinct values can be assumed by i^n?

1.2 Rectangular Representation

Just as a real number x may be represented by a point on a line, so may a complex number $z = (x, y)$ be represented by a point in the plane (Figure 1.1).

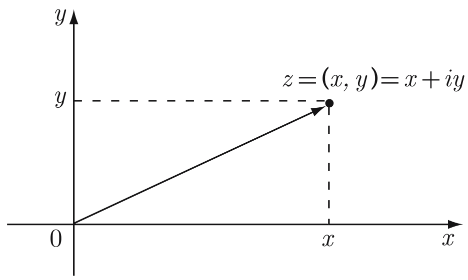

Figure 1.1. Cartesian representation of z in plane

Each complex number corresponds to one and only one point. Thus the terms complex number and point in the plane are used interchangeably. The x and y axes are referred to as the *real axis* and the *imaginary axis*, while the xy plane is called the *complex plane* or the z plane.

There is yet another interpretation of the complex numbers. Each point (x, y) of the complex plane determines a two-dimensional *vector* (directed line segment) from $(0, 0)$, the initial point, to (x, y), the terminal point. Thus the complex number may be represented by a vector. This seems natural in that the definition chosen for addition of complex numbers corresponds to vector addition; that is,

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2).$$

Geometrically, vector addition follows the so-called *parallelogram rule*, which we illustrate in Figure 1.2. From the point z_1, construct a vector equal in magnitude and direction to the vector z_2. The terminal point is the vector $z_1 + z_2$. Alternatively, if a vector equal in magnitude and direction to z_1 is joined to the vector z_2, the same terminal point is reached. This illustrates the *commutative property* of vector addition. Note that the vector $z_1 + z_2$ is a diagonal of the parallelogram formed. What would the other diagonal represent?

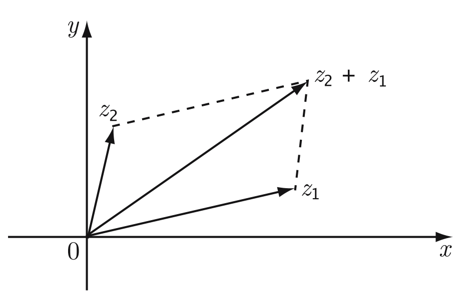

Figure 1.2. Illustration for parallelogram law

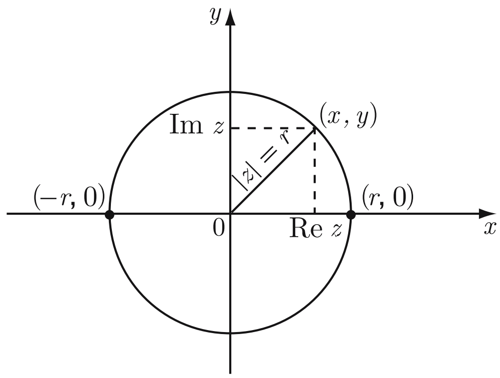

Figure 1.3. Modulus of a complex number z

By the *magnitude* (length) of the vector (x, y) we mean the distance of the point $z = (x, y)$ from the origin. This distance is called the *modulus* or *absolute value* of the complex number z, and denoted by $|z|$; its value is $\sqrt{x^2 + y^2}$. For each positive real number r, there are infinitely many distinct values (x, y) whose absolute value is $r = |z|$, namely the points on the circle $x^2 + y^2 = r^2$. Two of these points, $(r, 0)$ and $(-r, 0)$, are real numbers so that this definition agrees with the definition for the absolute value in the real field (see Figure 1.3).

Note that, for $z = (x, y)$,

$$\begin{cases} |x| = |\mathrm{Re}\, z| \leq |z|, \\ |y| = |\mathrm{Im}\, z| \leq |z|. \end{cases}$$

The distance between any two points $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$ is

$$|z_2 - z_1| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

The triangle inequalities

$$\begin{cases} |z_1 + z_2| \leq |z_1| + |z_2|, \\ |z_1 - z_2| \geq \big|\, |z_1| - |z_2| \,\big| \end{cases}$$

say, geometrically, that no side of a triangle is greater in length than the sum of the lengths of the other two sides, or less than the difference of the lengths of the other two sides (Figure 1.2). The algebraic verification of these inequalities is left to the reader.

Among all points whose absolute value is the same as that of $z = (x, y)$, there is one which plays a special role. The point $(x, -y)$ is called the *conjugate* of z and is denoted by $\overline{z}$. If we view the real axis as a two-way mirror, then $\overline{z}$ is the mirror image of z (Figure 1.4).

From the definitions we obtain the following properties of the conjugate:

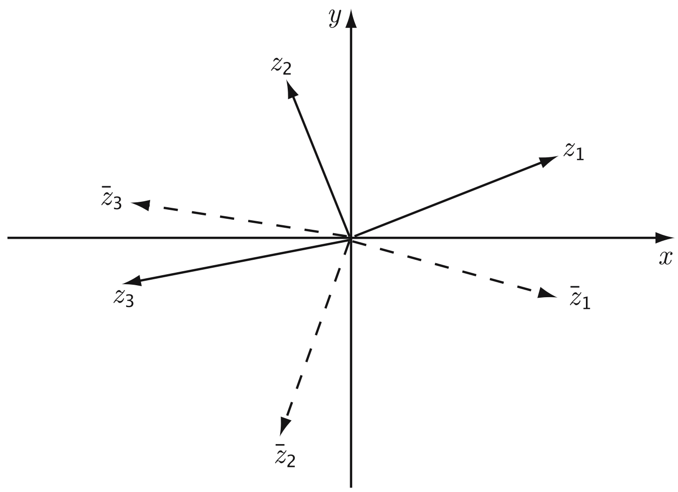

Figure 1.4. Mirror image of complex numbers

$$\begin{cases} \overline{z_1 + z_2} = \overline{z}_1 + \overline{z}_2, \\ \overline{z_1 z_2} = \overline{z}_1 \overline{z}_2. \end{cases} \tag{1.5}$$

Some of the important relationships between a complex number $z = (x, y)$ and its conjugates are

$$\begin{cases} z + \overline{z} = (2x, 0) = 2\mathrm{Re}\, z, \\ z - \overline{z} = (0, 2y) = 2i\mathrm{Im}\, z, \\ |z| = |\overline{z}| = \sqrt{x^2 + y^2}, \\ z\overline{z} = |z|^2. \end{cases} \tag{1.6}$$

The *squared* form of the absolute value in (1.6) is often the most workable. For example, to prove that the absolute value of the product of two complex numbers is equal to the product of their absolute values, we write

$$|z_1 z_2|^2 = (z_1 z_2)\overline{(z_1 z_2)} = (z_1 z_2)(\overline{z}_1 \overline{z}_2) = (z_1 \overline{z}_1)(z_2 \overline{z}_2) = (|z_1|\,|z_2|)^2.$$

Moreover, the conjugate furnishes us with a method of separating the inverse of a complex number into its real and imaginary parts:

$$(a + bi)^{-1} = \frac{1}{a + bi} \cdot \frac{\overline{a + bi}}{\overline{a + bi}} = \frac{a - bi}{a^2 + b^2} = \frac{a}{a^2 + b^2} - \frac{b}{a^2 + b^2} i.$$

Equation of a line in $\mathbb{C}$. Now we may rewrite the equation of a straight line in the plane, with the real and imaginary axes as axes of coordinates, as

$$ax + by + c = 0, \;\; a, b, c \in \mathbb{R}; \;\; \text{i.e.,}\; a\left(\frac{z + \overline{z}}{2}\right) + b\left(\frac{z - \overline{z}}{2i}\right) + c = 0,$$

where at least one of a, b is nonzero. That is,

$$(a - ib)z + (a + ib)\overline{z} + 2c = 0.$$

Conversely, by retracing the steps above, we see that

$$\alpha z + \beta \overline{z} + \gamma = 0 \tag{1.7}$$

represents a straight line provided $\alpha = \overline{\beta}$, $\alpha \neq 0$ and γ is real.

Equation of a circle in $\mathbb{C}$. A circle in $\mathbb{C}$ is the set of all point equidistant from a given point, the center. The standard equation of a circle in the xy plane with center at (a, b) and radius $r > 0$ is $(x - a)^2 + (y - b)^2 = r^2$. If we transform this by means of the substitution $z = x + iy,\ z_0 = a + ib$, then we have $z - z_0 = (x - a) + i(y - b)$ so that

$$(z - z_0)(\overline{z - z_0}) = |z - z_0|^2 = (x - a)^2 + (y - b)^2 = r^2.$$

Therefore, the equation of the circle in the complex form with center z_0 and radius r is $|z - z_0| = r$. In complex notation we may rewrite this as

$$z\overline{z} - (z\overline{z}_0 + \overline{z}z_0) + z_0\overline{z}_0 = r^2, \quad \text{i.e.} \quad z\overline{z} - 2\mathrm{Re}\,[z(a - ib)] + a^2 + b^2 - r^2 = 0,$$

where $z_0 = a + ib$. Thus, in general, writing $a - ib = \beta$ and $\gamma = a^2 + b^2 - r^2$, we see that

$$\alpha |z|^2 + \beta z + \overline{\beta}\overline{z} + \gamma = 0, \quad \text{i.e.} \quad \left| z + \frac{\beta}{\alpha} \right|^2 = \frac{|\beta|^2 - \alpha\gamma}{\alpha^2}, \tag{1.8}$$

represents a circle provided α, γ are real, $\alpha \neq 0$ and $|\beta|^2 - \alpha\gamma > 0$.

The formulas in (1.6) produce

$$|z_1 + z_2|^2 = |z_1|^2 + 2\mathrm{Re}\,(z_1\overline{z}_2) + |z_2|^2. \tag{1.9}$$

Also, for two complex numbers z_1 and z_2, we have

(i) $|1 - \overline{z}_1 z_2|^2 - |z_1 - z_2|^2 = (1 + |z_1|\,|z_2|)^2 - (|z_1| + |z_2|)^2$, since[2]

$$\begin{aligned}
\text{L.H.S} &= (1 - \overline{z}_1 z_2)(1 - z_1\overline{z}_2) - (z_1 - z_2)(\overline{z}_1 - \overline{z}_2) \\
&= 1 - (\overline{z}_1 z_2 + z_1\overline{z}_2) + |z_1 z_2|^2 \\
&\qquad - (|z_1|^2 + |z_2|^2 - z_1\overline{z}_2 - \overline{z}_1 z_2) \\
&= 1 + |z_1 z_2|^2 - (|z_1|^2 + |z_2|^2) \\
&= (1 - |z_1|^2)(1 - |z_2|^2) \\
&= \text{R.H.S.}
\end{aligned}$$

Further, it is also clear from (i) that if $|z_1| < 1$ and $|z_2| < 1$, then

[2] L.H.S is to mean left-hand side and R.H.S is to mean right-hand side.

$$|z_1 - z_2| < |1 - z_1\overline{z}_2|$$

and if either $|z_1| = 1$ or $|z_2| = 1$, then

$$|z_1 - z_2| = |1 - \overline{z}_1 z_2|.$$

(ii) $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2(|z_1|^2 + |z_2|^2)$ (*Parallelogram identity*); for,

$$\begin{aligned}\text{L.H.S} &= (z_1 + z_2)(\overline{z}_1 + \overline{z}_2) + (z_1 - z_2)(\overline{z}_1 - \overline{z}_2)\\ &= [|z_1|^2 + (z_1\overline{z}_2 + \overline{z}_1 z_2) + |z_2|^2]\\ &\qquad + [|z_1|^2 - (z_1\overline{z}_2 + \overline{z}_1 z_2) + |z_2|^2]\\ &= \text{R.H.S.}\end{aligned}$$

Example 1.3. Let us use the triangle inequality to find upper and lower bounds for $|z^4 - 3z + 1|^{-1}$ whenever $|z| = 2$. To do this, we need to find m and M so that $m \leq |z^4 - 3z + 1|^{-1} \leq M$ for $|z| = 2$. As $|3z - 1| \leq 3|z| + 1 = 7$ for $|z| = 2$, we have

$$|z^4 - 3z + 1| \geq |\,|z^4| - |3z - 1|\,| \geq 2^4 - 7 = 9$$

and $|z^4 - 3z + 1| \leq |z|^4 + |3z - 1| = 2^4 + 7 = 23$. Thus, for $|z| = 2$, we have

$$\frac{1}{23} \leq |z^4 - 3z + 1|^{-1} \leq \frac{1}{9}.$$ ●

Example 1.4. Suppose that we wish to find all circles that are orthogonal to both $|z| = 1$ and $|z - 1| = 4$. To do this, we consider two circles:

$$C_1 = \{z : \, |z - \alpha_1| = r_1\}, \quad C_2 = \{z : \, |z - \alpha_2| = r_2\}.$$

These two circles are orthogonal to each other if (see Figure 1.5)

$$r_1^2 + r_2^2 = |\alpha_1 - \alpha_2|^2.$$

In view of this observation, the conditions for which a circle $|z - \alpha| = R$ is orthogonal to both $|z| = 1$ and $|z - 1| = 4$ are given by

$$1 + R^2 = |\alpha - 0|^2 \ \text{ and } \ 4^2 + R^2 = |\alpha - 1|^2 = 1 + |\alpha|^2 - 2\text{Re}\,\alpha$$

which give $R = (|\alpha|^2 - 1)^{1/2}$ and $\text{Re}\,\alpha = -7$. Consequently,

$$\alpha = -7 + ib \ \text{ and } \ R = (49 + b^2 - 1)^{1/2} = (48 + b^2)^{1/2}$$

and the desired circles are given by

$$C_b : \, |z - (-7 + ib)| = (48 + b^2)^{1/2}, \quad b \in \mathbb{R}.$$ ●

Example 1.5. We wish to show that triangle $\triangle ABC$ with vertices z_1, z_2, z_3 is equilateral if and only if

$$z_1^2 + z_2^2 + z_3^2 = z_1 z_2 + z_2 z_3 + z_3 z_1. \tag{1.10}$$

To do this, we let $\alpha = z_2 - z_1$, $\beta = z_3 - z_2$, and $\gamma = z_1 - z_3$ so that $\alpha + \beta + \gamma = 0$. Further, if $\triangle ABC$ is equilateral, then (see Figure 1.6)

$$\begin{aligned}
\alpha + \beta + \gamma = 0 &\Longleftrightarrow \overline{\alpha} + \overline{\beta} + \overline{\gamma} = 0 \\
&\Longleftrightarrow \frac{\alpha\overline{\alpha}}{\alpha} + \frac{\beta\overline{\beta}}{\beta} + \frac{\gamma\overline{\gamma}}{\gamma} = 0 \\
&\Longleftrightarrow \frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = 0 \quad (\because |\alpha| = |\beta| = |\gamma|) \\
&\Longleftrightarrow \frac{1}{z_2 - z_1} + \frac{1}{z_3 - z_2} + \frac{1}{z_1 - z_3} = 0 \\
&\Longleftrightarrow (z_3 - z_2)(z_1 - z_3) + (z_2 - z_1)(z_1 - z_3) \\
&\qquad\qquad + (z_2 - z_1)(z_3 - z_2) = 0 \\
&\Longleftrightarrow z_1^2 + z_2^2 + z_3^2 = z_1 z_2 + z_2 z_3 + z_3 z_1.
\end{aligned}$$

Conversely, suppose that (1.10) holds. Then

$$\begin{aligned}
\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = 0 &\Longrightarrow \alpha\beta + \beta\gamma + \gamma\alpha = 0 \\
&\Longrightarrow \alpha\beta + \gamma(-\gamma) = 0, \quad \text{since } \alpha + \beta = -\gamma, \\
&\Longrightarrow \alpha\beta = \gamma^2.
\end{aligned}$$

Thus, $\alpha\beta = \gamma^2$. Similarly, $\beta\gamma = \alpha^2$ and $\gamma\alpha = \beta^2$. Further,

$$(\alpha\beta)(\overline{\alpha\beta}) = \gamma^2(\overline{\gamma})^2, \quad \text{i.e.,} \quad (\alpha\overline{\alpha})(\beta\overline{\beta})(\gamma\overline{\gamma}) = (\gamma\overline{\gamma})^3.$$

Because of the symmetry, we also have

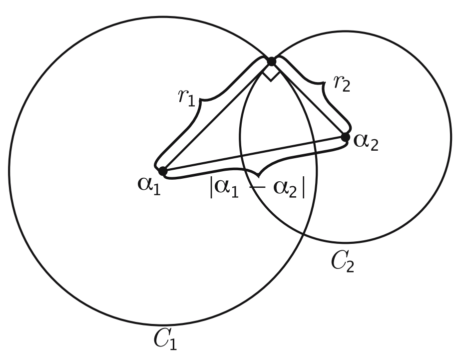

Figure 1.5. Orthogonal circles

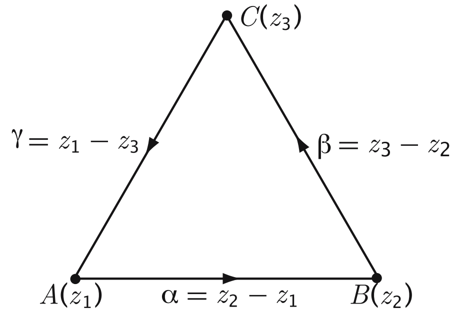

Figure 1.6. Equilateral triangle $\triangle ABC$

$$(\alpha\overline{\alpha})(\beta\overline{\beta})(\gamma\overline{\gamma}) = (\alpha\overline{\alpha})^3 \quad\text{and}\quad (\alpha\overline{\alpha})(\beta\overline{\beta})(\gamma\overline{\gamma}) = (\beta\overline{\beta})^3.$$

Thus,

$$\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = 0 \implies |\alpha|^3 = |\beta|^3 = |\gamma|^3 \implies |\alpha| = |\beta| = |\gamma|,$$

showing that $\triangle ABC$ is equilateral.

Here is an alternate proof. First we remark that equilateral triangles are preserved under linear transformations $f(z) = az + b$, which can be easily verified by replacing z_j by $az_j + b$ $(j = 1, 2, 3)$ in (1.10). By a suitable transformation, we can reduce the problem to a simpler one. If z_1, z_2, z_3 are the vertices of a degenerated equilateral triangle (i.e., $z_1 = z_2 = z_3$), then (1.10) holds. If two of the vertices are distinct, then, by a suitable transformation, we can take $z_1 = 0$ and $z_2 = 1$. Then (1.10) takes the form $1 + z_3^2 = z_3$, which gives

$$z_3 = \frac{1 + i\sqrt{3}}{2} \quad\text{or}\quad \frac{1 - i\sqrt{3}}{2}.$$

In either case $\{0, 1, z_3\}$ forms vertices of an equilateral triangle. ●

Example 1.6. Suppose we wish to describe geometrically the set S given by

$$S = \{z : |z - a| - |z + a| = 2c\} \quad (0 \neq a \in \mathbb{C},\ c \geq 0), \tag{1.11}$$

for the following situations:

(i) $c > |a|$ (ii) $c = 0$ (iii) $0 < c < a$ (iv) $c = a > 0$.

The triangle inequality gives that

$$|2a| = |z - a - (z + a)| \geq |z - a| - |z + a| = 2c, \quad\text{i.e., } c \leq |a|.$$

Thus, there are no complex numbers satisfying (1.11) if $c > |a|$. Hence, $S = \emptyset$ whenever $c > |a|$.

If $c = 0$, we have $|z - a| = |z + a|$ which shows that S is the line that is the perpendicular bisector of the line joining a and $-a$.

Next, we consider the case $a > c > 0$. Then, writing $z = x + iy$,

$$\begin{aligned} |z-a| - |z+a| = 2c &\iff |z-a|^2 = (2c + |z+a|)^2 \\ &\iff |z-a|^2 = 4c^2 + |z+a|^2 + 4c|z+a| \\ &\iff c|z+a| + c^2 = -a\mathrm{Re}\, z \quad (\mathrm{Re}\, z < 0) \\ &\iff c^2[|z|^2 + a^2 + 2a\mathrm{Re}\, z] = (c^2 + a\mathrm{Re}\, z)^2 \\ &\iff c^2|z|^2 - a^2(\mathrm{Re}\, z)^2 = c^2(c^2 - a^2) \\ &\iff \frac{x^2}{c^2} - \frac{y^2}{a^2 - c^2} = 1. \end{aligned}$$

Further, we observe that for $|z - a| - |z + a|$ to be positive, we must have $\mathrm{Re}\, z < 0$. Thus, if $a > c > 0$ we have

$$S = \left\{ x + iy : \frac{x^2}{c^2} - \frac{y^2}{a^2 - c^2} = 1 \right\}$$

and so S describes a hyperbola with focii at $a, -a$.

Finally, if $c = a$ then

$$|z-a| - |z+a| = 2a \iff |z+a| = -\mathrm{Re}\,(z+a) \implies \mathrm{Re}\,(z+a) < 0$$

and therefore, S in this case is the interval $(-\infty, -a]$. ●

Questions 1.7.

1. In Figure 1.2, would we still have a parallelogram if the vector z_2 were in the same or the opposite direction as that of z_1?
2. Geometrically, can we predict the quadrant of z_1+z_2 from our knowledge of z_1 and z_2?
3. Why don't we define multiplication of complex numbers as vector multiplication?
4. When does the triangle inequality become an equality?
5. What would be the geometric interpretation of the inequality for the sum of n complex numbers?
6. Name some interesting relationships between the points (x, y) and $(-x, y)$.
7. If a and b are positive rational numbers, why might we want to call the numbers $\sqrt{a} + \sqrt{b}$ and $\sqrt{a} - \sqrt{b}$ real conjugates?
8. Is every rational number algebraic? Are $\sqrt{3}$ and $\sqrt[5]{5} - 3i$ algebraic?
   Note: A number is algebraic if it is a solution of a polynomial (in z) with integer coefficients. Numbers which are not algebraic are called transcendental numbers.
9. What does $|z|^2 + \beta z + \overline{\beta}\overline{z} + \gamma = 0$ represent if $|\beta|^2 \geq \gamma$?
10. Is $|z + 1| + |z - 1| \leq 2\sqrt{2}$ if $|z| \leq 1$?

Exercises 1.8.

1. If $z_1 = 3 - 4i$ and $z_2 = -2 + 3i$, obtain graphically and analytically
(a) $2z_1 + 4z_2$ (b) $3z_1 - 2\overline{z}_2$ (c) $z_1 - \overline{z}_2 - 4$
(d) $|z_1 + z_2|$ (e) $|z_1 - z_2|$ (f) $|2\overline{z}_1 + 3\overline{z}_2 - 1|$.
2. Let $z_1 = x_1 + iay_1$ and $z_2 = x_2 - ib/y_1$, where a, b are real. Determine a condition on y_1 so that $z_1^{-1} + z_2^{-1}$ is real.
3. Identify all the points in the complex plane that satisfy the following relations.
(a) $1 < |z| \leq 3$ (b) $|(z-3)/(z+3)| < 2$
(c) $|z-1| + |z+1| = 2$ (d) $\mathrm{Re}\,(z-5) = |z| + 5$
(e) $\mathrm{Re}\,z^2 > 0$ (f) $\mathrm{Im}\,z^2 > 0$
(g) $\mathrm{Re}\,((1-i)z) = 2$ (h) $|z - i| = \mathrm{Re}\,z$
(i) $\mathrm{Re}\,(z) = |z|$ (j) $\mathrm{Re}\,(z^2) = 1$
(k) $\overline{z} = 5/(z-1)$ $(z \neq 1)$ (l) $[\mathrm{Im}\,(iz)]^2 = 1$.
4. Let $|(z-a)/(z-b)| = M$, where a and b are complex constants and $M > 0$. Describe this curve and explain what happens as $M \to 0$ and as $M \to \infty$.
5. Find a complex form for the hyperbola with real equation $9x^2 - 4y^2 = 36$.
6. If $|z| < 1$, prove that
$$\text{(a) } \mathrm{Re}\left(\frac{1}{1-z}\right) > \frac{1}{2} \quad \text{(b) } \mathrm{Re}\left(\frac{z}{1-z}\right) > -\frac{1}{2} \quad \text{(c) } \mathrm{Re}\left(\frac{1+z}{1-z}\right) > 0.$$
7. If $P(z)$ is a polynomial equation with real coefficients, show that z_1 is a root if and only if $\overline{z}_1$ is a root. Conclude that any polynomial equation of odd degree with real coefficients must have at least one real root. Can you prove this using elementary calculus?
8. Prove that, for every $n \geq 1$,
$$|z_1 + z_2 + \cdots + z_n| \leq |z_1| + |z_2| + \cdots + |z_n|.$$
9. Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be complex numbers. Prove the Schwarz inequality,
$$\left|\sum_{k=1}^{n} a_k b_k\right|^2 \leq \left(\sum_{k=1}^{n} |a_k|^2\right)\left(\sum_{k=1}^{n} |b_k|^2|\right).$$
When will equality hold?
10. Define $e(\alpha) = \cos\alpha + i\sin\alpha$, for α real. Prove the following.
(a) $e(0) = 1$ (b) $|e(\alpha)| = 1$
(c) $e(\alpha_1 + \alpha_2) = e(\alpha_1)e(\alpha_2)$ (d) $e(n\alpha) = [e(\alpha)]^n$.

Which of these properties does the real-valued function $f(x) = e^x$ satisfy?

11. Show that the line connecting the complex numbers z_1 and z_2 is perpendicular to the line connecting z_3 and z_4 if and only if

$$\mathrm{Re}\,\{(z_1 - z_2)(\overline{z}_3 - \overline{z}_4)\} = 0.$$

12. If a, b are real numbers in the unit interval $(0,1)$, then when do the three points $z_1 = a + i$, $z_2 = 1 + ib$ and $z_3 = 0$ form an equilateral triangle?
13. If $|z_j| = 1$ $(j = 1, 2, 3)$ such that $z_1 + z_2 + z_3 = 0$, then show that z_j's are the vertices of an equilateral triangle.

1.3 Polar Representation

In Section 1.2, the magnitude of the vector $z = x + iy$ was discussed. What about its direction? A measurement of the angle θ that the vector $z\,(\neq 0)$ makes with the positive real axis is called an *argument* of z (see Figure 1.7). Thus, we may express the point $z = (x, y)$ in the "new" form

$$(r\cos\theta, r\sin\theta).$$

This, of course, is just the polar coordinate representation for the complex number z. We have the familiar relations

$$r = |z| = \sqrt{x^2 + y^2} \quad \text{and} \quad \tan\theta = \frac{y}{x}.$$

The real numbers r and θ, like x and y, uniquely determine the complex number z. Unfortunately, the converse isn't completely true. While z uniquely determines the x and y, hence r, the value of θ is determined up to a multiple of 2π. There are infinitely many distinct arguments for a given complex number z, and the symbol $\arg z$ is used to indicate *any one* of them. Thus the arguments of the complex number (2, 2) are

$$\frac{\pi}{4} + 2k\pi \quad (k = 0, \pm 1, \pm 2, \dots).$$

This inconvenience can sometimes (although not always) be ignored by distinguishing (arbitrarily) one particular value of $\arg z$. We use the symbol $\mathrm{Arg}\, z$ to stand for the unique determination of θ for which $-\pi < \arg z \leq \pi$. This θ is called the *principal value* of the argument. To illustrate,

$$\mathrm{Arg}\,(2,2) = \frac{\pi}{4}, \quad \mathrm{Arg}\,(0,-5) = -\frac{\pi}{2}, \quad \mathrm{Arg}\,(-1, \sqrt{3}) = \frac{2\pi}{3}.$$

Note that $\mathrm{Re}\, z > 0$ is equivalent to $|\mathrm{Arg}\, z| < \pi/2$. If $x = y = 0$, the expression $\tan\theta = y/x$ has no meaning. For this reason, $\arg z$ is not defined when $z = 0$.

Suppose that z_1 and z_2 have the polar representations

$$z_1 = r_1(\cos\theta_1 + i\sin\theta_1) \quad \text{and} \quad z_2 = r_2(\cos\theta_2 + i\sin\theta_2).$$

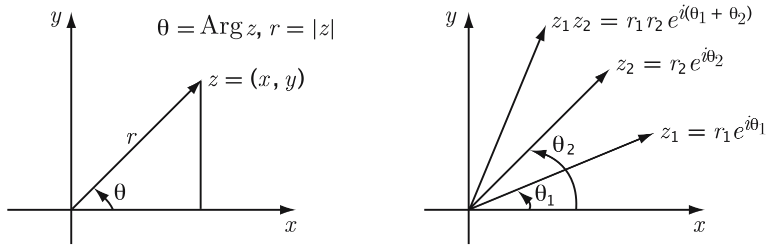

Figure 1.7. Polar representation of z and z_1z_2

Then

$$\begin{aligned} z_1z_2 &= r_1r_2[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \sin\theta_2\cos\theta_1)] \\ &= r_1r_2[\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)]. \end{aligned}$$

Loosely speaking, we may say that the argument of the product of two nonzero complex numbers is equal to the sum of their arguments; that is,

$$\arg(z_1z_2) = \arg z_1 + \arg z_2. \tag{1.12}$$

We understand (1.12) to mean that if θ_1 is one of the values of $\arg z_1$ and θ_2 is one of the values of $\arg z_2$, then $\theta_1 + \theta_2$ is one of the values of $\arg(z_1z_2)$. Since (1.12) is valid only up to a multiple of 2π, a more explicit formulation is

$$\arg z_1z_2 = \arg z_1 + \arg z_2 + 2k\pi \quad (k \text{ an integer})$$

or

$$\arg z_1z_2 = \arg z_1 + \arg z_2 \pmod{2\pi}$$

(see Figure 1.7). To illustrate, we observe that if $z = (-1 + i\sqrt{3})/2$, then $z^2 = (-1 - i\sqrt{3})/2$ so that

$$\operatorname{Arg} z = \frac{2\pi}{3} \quad\text{and}\quad \operatorname{Arg}(z^2) = -\frac{2\pi}{3}.$$

Thus, $\operatorname{Arg}(z.z) = \operatorname{Arg} z + \operatorname{Arg} z - 2\pi$.

An induction argument (no pun intended) shows that if z_i has modulus r_i and argument θ_i $(i = 1, 2, \dots, n)$, then

$$\begin{aligned} z_1z_2\cdots z_n = r_1r_2\cdots r_n[&\cos(\theta_1+\theta_2+\cdots+\theta_n) \\ &+ i\sin(\theta_1+\theta_2+\cdots+\theta_n)]. \end{aligned} \tag{1.13}$$

Example 1.9. Let $z_1 = 1 + i$ and $z_2 = \sqrt{3} + i$. We wish to express them in polar form and then verify the identities that hold for multiplication and division of z_1 and z_2, respectively. To do this, we may write

$$z_1 = \sqrt{2}e^{i\pi/4} \quad\text{and}\quad z_2 = 2e^{i\pi/6}.$$

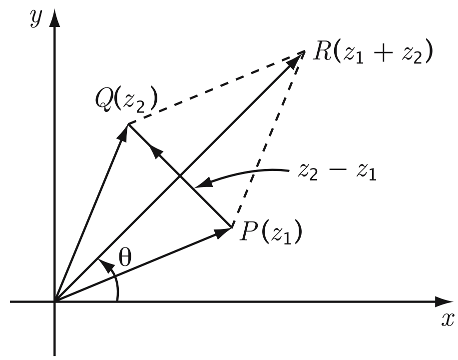

Figure 1.8. Geometric proof for Example 1.10

Then

$$z_1 z_2 = 2\sqrt{2}e^{i5\pi/12} \text{ and } \frac{z_1}{z_2} = \frac{1}{\sqrt{2}}e^{i\pi/12}.$$

Thus, in this particular problem of product and division, it follows that

$$\text{Arg}\,(z_1 z_2) = \text{Arg}\,z_1 + \text{Arg}\,z_2 \text{ and } \text{Arg}\left(\frac{z_1}{z_2}\right) = \text{Arg}\,z_1 - \text{Arg}\,z_2.$$

Similarly, we may easily check the following:

(i) $(1 - i\sqrt{3})/(1 + i\sqrt{3}) = e^{i\theta},\ \theta = 2\pi/3 + 2k\pi$;
(ii) $(-\sqrt{3} + i)(1 + i)/(1 + i\sqrt{3}) = \sqrt{2}e^{i\theta},\ \theta = 3\pi/4 + 2k\pi$;
(iii) $(1 - 3i)/(2 - i) = \sqrt{2}e^{i\theta},\ \theta = -\pi/4 + 2k\pi$,

where k is an integer. ●

Example 1.10. Suppose that z_1 and z_2 are two nonzero complex numbers such that $|z_1| = |z_2|$ but $z_1 \neq \pm z_2$. Then we wish to show that the quotient $(z_1 + z_2)/(z_1 - z_2)$ is a purely imaginary number. For a geometric proof, we consider the parallelogram $OPRQ$ shown in Figure 1.8. Since the sides OP and OQ are equal in length, $OPRQ$ is a rhombus. Thus, the vector $\overrightarrow{OR}$ is perpendicular to the vector $\overrightarrow{PQ}$, and so

$$\text{Arg}\,(z_1 + z_2) = \text{Arg}\,(z_1 - z_2) \pm i\pi/2.$$

For an analytic proof, we may rewrite

$$w = \frac{z_1 + z_2}{z_1 - z_2} = \frac{1 + z}{1 - z} \quad (z = z_2/z_1).$$

The hypotheses imply that $|z| = 1,\ z \neq \pm 1$. Therefore, letting $z = e^{i\theta}$ with $\theta \in (0, 2\pi)\backslash\{\pi\}$,

$$w = \frac{1 + e^{i\theta}}{1 - e^{i\theta}} = \frac{e^{-i\theta/2} + e^{i\theta/2}}{e^{-i\theta/2} - e^{i\theta/2}} = \frac{2\cos(\theta/2)}{-2i\sin(\theta/2)} = i\cot(\theta/2),$$

which is a purely imaginary number. ●

Example 1.11. Let $z = \sin\theta + i\cos 2\theta$ and $w = \cos\theta + i\sin 2\theta$. We wish to show that there exists no value of θ for which $z = w$. To do this, we first note that

$$z = w \iff \sin\theta = \cos\theta \quad \text{and} \quad \cos 2\theta = \sin 2\theta.$$

There exists no values of θ satisfying both conditions, because $\sin\theta = \cos\theta$ implies that $\cos 2\theta = \cos^2\theta - \sin^2\theta = 0$, and so the second condition reduces to $\sin 2\theta = 2\sin\theta\cos\theta = 0$, i.e., $\sin\theta = 0 = \cos\theta$. ●

Remark 1.12. Geometric considerations (Figures 1.2 and 1.7) indicate that the rectangular representation will frequently be more useful for problems involving sums of complex numbers, with polar representation being more useful for problems involving products. ●

If we let $z_1 = z_2 = \cdots = z_n$ in (1.13), we obtain

$$z^n = r^n(\cos n\theta + i\sin n\theta). \tag{1.14}$$

For $|z| = 1$ (the unit circle), (1.14) reduces to

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta, \tag{1.15}$$

a theorem of DeMoivre.

The possibility of finding nth roots of the complex number is suggested by (1.14). A complex number z is an nth root of z_0 if $z^n = z_0$, written $z = z_0^{1/n}$.

The problem is to reverse the multiplicative operation and determine a number which, when multiplied by itself n times, furnishes us with the original number. Given a complex number $z_0 = r_0(\cos\theta_0 + i\sin\theta_0)$, how do you find a complex number $z = r(\cos\theta + i\sin\theta)$ such that $z^n = z_0$? By (1.14), we must have

$$r^n(\cos n\theta + i\sin n\theta) = r_0(\cos\theta_0 + i\sin\theta_0). \tag{1.16}$$

Since $|\cos\alpha + i\sin\alpha| = 1$ for all real α, (1.16) yields the relations

$$r^n = r_0, \quad \cos n\theta + i\sin n\theta = \cos\theta_0 + i\sin\theta_0. \tag{1.17}$$

The first relation in (1.17) shows that $|z| = r_0^{1/n}$, which we already knew (why)? But the second gives important information about the argument of z, namely, that $n\arg z$ differs from $\arg z_0$ by a multiple of 2π (that is, $n\theta = \theta_0 + 2k\pi$, $k = 0, \pm1, \pm2, \ldots$):

$$\theta = \frac{\theta_0 + 2k\pi}{n}. \tag{1.18}$$

How many integers k in (1.18) produce distinct solutions? We have

$$z = z_0^{1/n} = r_0^{1/n}\left\{\cos\left(\frac{\theta_0 + 2k\pi}{n}\right) + i\sin\left(\frac{\theta_0 + 2k\pi}{n}\right)\right\}. \tag{1.19}$$

For each k ($k = 0, 1, 2, \ldots, n-1$), there is a different value for z. We leave it for the reader to verify that there are no more solutions. Thus, given $z_0 \neq 0$, there are exactly n distinct complex numbers z such that $z^n = z_0$.

By letting $z_0 = 1$ in (1.19), we may find the nth roots of unity. If $z^n = 1$, then

$$z = \cos\left(\frac{2k\pi}{n}\right) + i \sin\left(\frac{2k\pi}{n}\right) \quad (k = 0, 1, 2, \ldots, n-1). \tag{1.20}$$

Geometrically, the solutions represent the n vertices of a regular polygon of n sides inscribed in a circle with center at the origin and radius equal to one. See Figures 1.9 and 1.10 for the inscribed square and pentagon.

By (1.20), the difference in the arguments of any two successive nth roots of unity is constant ($2\pi/n$). If we let

$$\omega = \cos\frac{2\pi}{n} + i \sin\frac{2\pi}{n},$$

then each root of unity may be expressed as a multiple of ω; that is,

$$\omega, \omega^2, \omega^3, \ldots, \omega^{n-1}, \quad \omega^n = \omega^0 = 1.$$

This gives interesting information about the sums and products of the roots of the unity, namely, that the product of any two roots of unity is also a root of unity, and that the sum of all nth roots of unity is zero. The latter statement follows from the identify

$$1 + \omega + \omega^2 + \cdots + \omega^{n-1} = \frac{1-\omega^n}{1-\omega}.$$

Using (1.19), we easily see, for instance, the following:

(a) $*\sqrt{3+4i} = \pm(2+i)$
(b) $*\sqrt{-3+4i} = \pm(1+2i)$

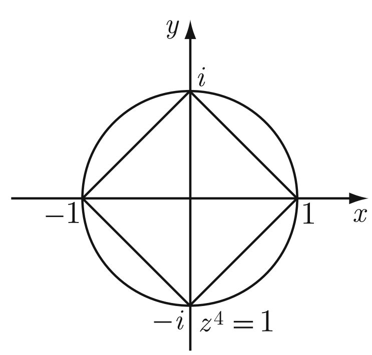

Figure 1.9. Illustration for the 4th roots of unity

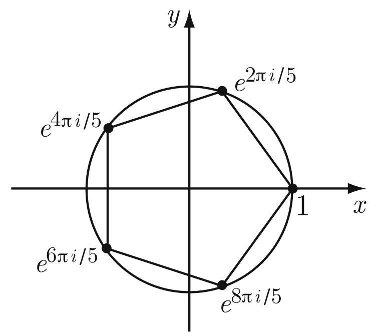

Figure 1.10. Illustration for the 5th roots of unity

(c) $*\sqrt{1+i} = \pm\left(\sqrt{\frac{\sqrt{2}+1}{2}} + i\sqrt{\frac{\sqrt{2}-1}{2}}\right)$
(d) $*\sqrt{2i} = \pm(1+i)$
(e) $*\sqrt{\frac{1-i\sqrt{3}}{2}} = \pm\left(\frac{\sqrt{3}-i}{2}\right)$
(f) $*\sqrt{1+i\sqrt{3}} = \pm\left(\frac{\sqrt{3}+i}{\sqrt{2}}\right)$
(g) $*\sqrt{-5-12i} = \pm(-2+3i)$
(h) $*\sqrt{5+12i} = \pm(3+2i)$
(i) $*\sqrt{-5+12i} = \pm(2+3i)$.

Here $*\sqrt{a+ib}$ denotes the two 2th roots of the complex number $a+ib$.

Since the n nth roots of unity are given by (1.20), we have

$$z^n - 1 = (z-1)(z-\omega_1)(z-\omega_2)\cdots(z-\omega_{n-1}), \quad \omega_k = \omega^k = e^{2\pi k i/n}.$$

Dividing both sides by $z-1$, using the identity

$$1 + z + z^2 + \cdots + z^{n-1} = \frac{1-z^n}{1-z} \quad (z \neq 1),$$

and letting $z \to 1$, we have

$$n = (1-\omega_1)(1-\omega_2)\cdots(1-\omega_{n-1}), \text{ and}$$
$$n = (1-\overline{\omega}_1)(1-\overline{\omega}_2)\cdots(1-\overline{\omega}_{n-1}).$$

As $(1-e^{-i\theta})(1-e^{i\theta}) = 2(1-\cos\theta) = 4\sin^2(\theta/2)$, it follows that

$$n^2 = \prod_{k=1}^{n-1} |1-\omega_k|^2 = \prod_{k=1}^{n-1}\left\{4\sin^2\left(\frac{k\pi}{n}\right)\right\} = 2^{2(n-1)}\prod_{k=1}^{n-1}\sin^2\left(\frac{k\pi}{n}\right).$$

Taking the positive square root on both sides we have

$$n = 2^{n-1} \prod_{k=1}^{n-1} \sin\left(\frac{k\pi}{n}\right), \quad n > 1. \tag{1.21}$$

We can make the following generalization: Consider the equation

$$M_a(z) = z^{2n} - 2z^n a^n \cos n\phi + a^{2n} = 0 \quad (n \in \mathbb{N},\ a \in \mathbb{R}^+,\ \phi \in \mathbb{R}).$$

Solving this for z^n, we find $z^n = a^n e^{\pm in\phi}$ so that

$$M_a(z) = [z^n - a^n e^{in\phi}][z^n - a^n e^{-in\phi}].$$

Therefore, using the concept of nth root of a complex number, we can write

$$\begin{aligned} M_a(z) &= \prod_{k=1}^{n} \left[z - ae^{i(\phi+2k\pi/n)}\right]\left[z - ae^{-i(\phi+2k\pi/n)}\right] \\ &= \prod_{k=1}^{n} \left[z^2 - 2za\cos\left(\phi + \frac{2k\pi}{n}\right) + a^2\right]. \end{aligned} \tag{1.22}$$

Some special cases of (1.22) follow:

(a) Taking $\phi = 0$, we have

$$(z^n - a^n)^2 = \prod_{k=1}^{n} \left[z^2 - 2za\cos\left(\frac{2k\pi}{n}\right) + a^2\right].$$

(b) Taking $\phi = \pi/n$, we have

$$(z^n + a^n)^2 = \prod_{k=1}^{n} \left[z^2 - 2za\cos\left(\frac{(2k+1)\pi}{n}\right) + a^2\right].$$

(c) If $a = 1$ then, on dividing (1.22) by z^n, $z \neq 0$, we have

$$z^n + z^{-n} - 2\cos(n\phi) = \prod_{k=1}^{n} \left[z + z^{-1} - 2\cos\left(\phi + \frac{2k\pi}{n}\right)\right]$$

and so, if $z = e^{i\theta}$, this becomes

$$\cos(n\theta) - \cos(n\phi) = 2^{n-1} \prod_{k=1}^{n} \left[\cos\theta - \cos\left(\phi + \frac{2k\pi}{n}\right)\right]$$

which is, for $\cos\theta \neq \cos\phi$, equivalent to

$$\frac{\cos(n\theta) - \cos(n\phi)}{\cos\theta - \cos\phi} = 2^{n-1} \prod_{k=1}^{n-1} \left[\cos\theta - \cos\left(\phi + \frac{2k\pi}{n}\right)\right].$$

In the limiting case when $\theta, \phi \to 0$, the above reduces to

$$n = 2^{n-1} \prod_{k=1}^{n-1} \sin\left(\frac{k\pi}{n}\right),$$

which is nothing but (1.21).

Questions 1.13.

1. What problem would be created by defining the argument of $z = 0$ to be zero?
2. Loosely speaking, for complex numbers z_1 and z_2 we have
$$\arg(z_1 z_2) = \arg z_1 + \arg z_2.$$
What real-valued functions have the property that
$$f(x_1 x_2) = f(x_1) + f(x_2)?$$
3. When does $\operatorname{Arg}(z_1 z_2) = \operatorname{Arg} z_1 + \operatorname{Arg} z_2$?
4. How are the complex numbers z_1 and z_2 related if $\arg(z_1) = \arg z_2$?
5. How are the arguments $\arg(z_1)$ and $\arg z_2$ related if $z_1 = z_2$?
6. How are the arguments $\arg(z_1)$ and $\arg z_2$ related if $\operatorname{Re}(z_1 \overline{z}_2) = |z_1 z_2|$?
7. How are the arguments $\arg(z_1)$ and $\arg z_2$ related if $|z_1+z_2| = |z_1|+|z_2|$?
8. As the complex number z approaches the negative real axis from above and below, what is happening to $\operatorname{Arg} z$? What if z approaches the *positive* real axis from above and below?
9. How do the arguments of z and $1/z$ compare?
10. How do the arguments of z and $\overline{z}$ compare?
11. How do the arguments of $\overline{z}$ and $1/z$ compare?
12. What is the position of the complex number $(\cos\alpha + i\sin\alpha)z$ relative to the position of z?
13. What are some differences between the terms angle, real number, and argument?
14. Of what use might the binomial theorem be in this section?
15. For which integers n does $z^n = 1$ have only real solutions?
16. For which complex numbers z does $\sqrt{z/\overline{z}} = z/|z|$?
17. Is it always the case that for any given nonzero complex number, either $\sqrt{z^2} = z$ or $\sqrt{z^2} = -z$?
18. Which postulates for a field are satisfied by the roots of unity under ordinary addition and multiplication of complex numbers?
19. What can you say about the nth roots of an arbitrary complex number?
20. For α an arbitrary real number, how many solutions might you expect $z^\alpha = 1$ to have?
21. If $z = e^{i\alpha}$ $(\alpha \in (0, 2\pi))$, is $(1+z)/(1-z)$ equal to $i\cot(\alpha/2)$?

Exercises 1.14.

1. For a fixed positive integer n, determine the real part of $(1+i\sqrt{3})^n$.
2. Find two complex numbers z_1 and z_2 so that
$$\operatorname{Arg}(z_1 z_2) \neq \operatorname{Arg} z_1 + \operatorname{Arg} z_2.$$
3. Find two complex numbers z_1 and z_2 so that
$$\operatorname{Arg}(z_1 z_2) = \operatorname{Arg} z_1 + \operatorname{Arg} z_2.$$
4. Describe the following regions geometrically.
(a) $\operatorname{Arg} z = \pi/6, \quad |z| > 1$ (b) $\pi/4 < \operatorname{Arg} z < \pi/2$
(c) $-\pi < \operatorname{Arg} z < 0, \quad |z+i| > 2$ (d) $1 < |z-1| < 5$.
5. If $|1-z| < 1$, show that $|\operatorname{Arg} z| < \pi/2$.
6. If $|z| < 1$, show that $|\operatorname{Arg}((1+z)/(1-z))| < \pi/2$.
7. If $\operatorname{Re} z > 0$, show that $\operatorname{Re}(1/z) > 0$. If $\operatorname{Re} z > a > 0$, what can you say about $\operatorname{Re}(1/z)$?
8. If $|z| = 1$, $z \neq -1$, show that z may be expressed in the form
$$z = \frac{1+it}{1-it},$$
where t is a real number.
9. Write the polar form of the following:
(a) $\dfrac{1+\cos\phi + i\sin\phi}{1+\cos\phi - i\sin\phi} \quad (0 < \phi < \pi/2)$
(b) $\dfrac{1+\cos\phi + i\sin\phi}{1-\cos\phi - i\sin\phi}$
(c) $1-\sin\phi + i\cos\phi \quad (0 < \phi < \pi/2)$
(d) $-\sin\phi - i\cos\phi$
(e) $(1+i)^n \quad (n \in \mathbb{N})$
(f) $(1+i\sqrt{3})^n + (1-i\sqrt{3})^n \quad (n \in \mathbb{N})$.
10. Find all values of the following and simplify the expressions as much as possible.
(a) $i^{1/2}$ (b) $i^{1/4}$ (c) $(-i)^{1/3}$ (d) $\sqrt{1+i}$
(e) $\sqrt[6]{8}$ (f) $\sqrt{4+3i}$ (g) $(4-3i)^{1/3}$ (h) $\sqrt{2+i}$
11. If $\omega = (-1+i\sqrt{3})/2$ is a cube root of unity and if
$$S_n = 1 - \omega + \omega^2 + \cdots + (-1)^{n-1}\omega^{n-1},$$
then find a formula for S_n.
12. Let ω be a cube root of unity and let a, b, c be real. Determine a condition on a, b, c so that $(a + b\omega + c\omega^2)^3$ is real.
13. Let ω be a cube root of unity. Determine the value of
(a) $(1+\omega)^3$ (b) $(1+2\omega+\omega^2)(1+\omega+2\omega^2)$
(c) $(1+\omega+2\omega^2)^9$ (d) $(1+3\omega+2\omega^2)(1+4\omega+3\omega^2)$.

14. Let $\omega \neq 1$ be an nth root of unity. Show that

$$1 + 2\omega + 3\omega^2 + \cdots + n\omega^{n-1} = -\frac{n}{1-\omega}.$$

15. Let $\omega_k = \cos(2k\pi/n) + i\sin(2k\pi/n)$. Show that $\sum_{k=1}^{n} |\omega_k - \omega_{k-1}| < 2\pi$ for all values of n. What happens as n approaches ∞?
16. Find the roots of the equation $(1+z)^5 = (1-z)^5$.
17. Find α, β, γ and δ such that the roots of the equation

$$z^5 + \alpha z^4 + \beta z^3 + \gamma z^2 + \delta z + \eta = 0$$

lie on a regular pentagon centered at 1.
18. Prove that for any real x and a natural number n,

$$e^{i2n\cot^{-1}(x)} \left(\frac{ix+1}{ix-1} \right)^n = 1.$$

19. Find a positive integer n such that

(i) $(\sqrt{3}+i)^n = 2^n$ (ii) $(-1+i)^n = 2^{n/2}$.