12.11 Laplace's Equation in Cylindrical and Spherical Coordinates. Potential

One of the most important PDEs in physics and engineering applications is **Laplace's equation**, given by

$$\nabla^2 u = u_{xx} + u_{yy} + u_{zz} = 0. \tag{1}$$

Here, x, y, z are Cartesian coordinates in space (Fig. 167 in Sec. 9.1), $u_{xx} = \partial^2 u/\partial x^2$, etc. The expression $\nabla^2 u$ is called the **Laplacian** of u. The theory of the solutions of (1) is called **potential theory**. Solutions of (1) that have ***continuous*** second partial derivatives are known as **harmonic functions**.

Laplace's equation occurs mainly in **gravitation**, **electrostatics** (see Theorem 3, Sec. 9.7), steady-state **heat flow** (Sec. 12.5), and **fluid flow** (to be discussed in Sec. 18.4).

Recall from Sec. 9.7 that the gravitational **potential** $u(x, y, z)$ at a point (x, y, z) resulting from a single mass located at a point (X, Y, Z) is

$$u(x, y, z) = \frac{c}{r} = \frac{c}{\sqrt{(x - X)^2 + (y - Y)^2 + (z - Z)^2}} \qquad (r > 0) \tag{2}$$

and u satisfies (1). Similarly, if mass is distributed in a region T in space with density $\rho(X, Y, Z)$, its potential at a point (x, y, z) not occupied by mass is

$$u(x, y, z) = k \iiint_T \frac{\rho(X, Y, Z)}{r}\, dX\, dY\, dZ. \tag{3}$$

It satisfies (1) because $\nabla^2(1/r) = 0$ (Sec. 9.7) and ρ is not a function of x, y, z.

Practical problems involving Laplace's equation are boundary value problems in a region T in space with boundary surface S. Such problems can be grouped into three types (see also Sec. 12.6 for the two-dimensional case):

(I) **First boundary value problem or Dirichlet problem** if u is prescribed on S.

(II) **Second boundary value problem** or **Neumann problem** if the normal derivative $u_n = \partial u/\partial n$ is prescribed on S.

(III) **Third** or **mixed boundary value problem** or **Robin problem** if u is prescribed on a portion of S and u_n on the remaining portion of S.

In general, when we want to solve a boundary value problem, we have to first select the appropriate coordinates in which the boundary surface S has a simple representation. Here are some examples followed by some applications.

Laplacian in Cylindrical Coordinates

The first step in solving a boundary value problem is generally the introduction of coordinates in which the boundary surface S has a simple representation. Cylindrical symmetry (a cylinder as a region T) calls for cylindrical coordinates r, θ, z related to x, y, z by

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad z = z \qquad \text{(Fig. 311).} \tag{4}$$

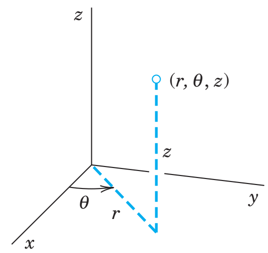

Fig. 311. Cylindrical coordinates ($r \geqq 0, 0 \leqq \theta \leqq 2\pi$)

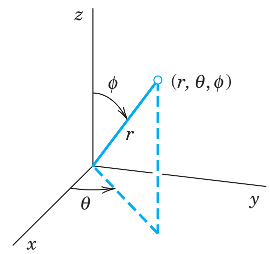

Fig. 312. Spherical coordinates ($r \geqq 0, 0 \leqq \theta \leqq 2\pi, 0 \leqq \varphi \leqq \pi$)

For these we get $\nabla^2 u$ immediately by adding u_{zz} to (5) in Sec. 12.10; thus,

$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2}. \tag{5}$$

Laplacian in Spherical Coordinates

Spherical symmetry (a ball as region T bounded by a sphere S) requires **spherical coordinates** r, θ, ϕ related to x, y, z by

$$x = r\cos\theta\,\sin\phi, \qquad y = r\sin\theta\,\sin\phi, \qquad z = r\cos\phi \qquad \text{(Fig. 312).} \tag{6}$$

Using the chain rule (as in Sec. 12.10), we obtain $\nabla^2 u$ in spherical coordinates

$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \phi^2} + \frac{\cot\phi}{r^2}\frac{\partial u}{\partial \phi} + \frac{1}{r^2\sin^2\phi}\frac{\partial^2 u}{\partial \theta^2}. \tag{7}$$

We leave the details as an exercise. It is sometimes practical to write (7) in the form

$$\nabla^2 u = \frac{1}{r^2}\left[\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{\sin\phi}\frac{\partial}{\partial \phi}\left(\sin\phi\,\frac{\partial u}{\partial \phi}\right) + \frac{1}{\sin^2\phi}\frac{\partial^2 u}{\partial \theta^2}\right]. \tag{7$'$}$$

Remark on Notation. Equation (6) is used in calculus and extends the familiar notation for polar coordinates. Unfortunately, some books use θ and ϕ interchanged, an extension of the notation $x = r\cos\phi$, $y = r\sin\phi$ for polar coordinates (used in some European countries).

Boundary Value Problem in Spherical Coordinates

We shall solve the following **Dirichlet problem** in spherical coordinates:

$$\nabla^2 u = \frac{1}{r^2}\left[\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{\sin\phi}\frac{\partial}{\partial \phi}\left(\sin\phi\,\frac{\partial u}{\partial \phi}\right)\right] = 0. \tag{8}$$

$$u(R, \phi) = f(\phi) \tag{9}$$

$$\lim_{r\to\infty} u(r, \phi) = 0. \tag{10}$$

The PDE (8) follows from (7) or (7′) by assuming that the solution u will not depend on θ because the Dirichlet condition (9) is independent of θ. This may be an electrostatic potential (or a temperature) $f(\phi)$ at which the sphere S: $r = R$ is kept. Condition (10) means that the potential at infinity will be zero.

Separating Variables by substituting $u(r, \phi) = G(r)H(\phi)$ into (8). Multiplying (8) by r^2, making the substitution and then dividing by GH, we obtain

$$\frac{1}{G}\frac{d}{dr}\left(r^2\frac{dG}{dr}\right) = -\frac{1}{H\sin\phi}\frac{d}{d\phi}\left(\sin\phi\frac{dH}{d\phi}\right).$$

By the usual argument both sides must be equal to a constant k. Thus we get the two ODEs

$$\frac{1}{G}\frac{d}{dr}\left(r^2\frac{dG}{dr}\right) = k \qquad \text{or} \qquad r^2\frac{d^2G}{dr^2} + 2r\frac{dG}{dr} = kG \tag{11}$$

and

$$\frac{1}{\sin\phi}\frac{d}{d\phi}\left(\sin\phi\frac{dH}{d\phi}\right) + kH = 0. \tag{12}$$

The solutions of (11) will take a simple form if we set $k = n(n+1)$. Then, writing $G' = dG/dr$, etc., we obtain

$$r^2G'' + 2rG' - n(n+1)\,G = 0. \tag{13}$$

This is an **Euler–Cauchy equation**. From Sec. 2.5 we know that it has solutions $G = r^a$. Substituting this and dropping the common factor r^a gives

$$a(a-1) + 2a - n(n+1) = 0. \qquad \text{The roots are} \qquad a = n \quad \text{and} \quad -n-1.$$

Hence solutions are

$$G_n(r) = r^n \qquad \text{and} \qquad G_n^*(r) = \frac{1}{r^{n+1}}. \tag{14}$$

We now solve (12). Setting $\cos\phi = w$, we have $\sin^2\phi = 1 - w^2$ and

$$\frac{d}{d\phi} = \frac{d}{dw}\frac{dw}{d\phi} = -\sin\phi\frac{d}{dw}.$$

Consequently, (12) with $k = n(n+1)$ takes the form

$$\frac{d}{dw}\left[(1-w^2)\frac{dH}{dw}\right] + n(n+1)H = 0. \tag{15}$$

This is **Legendre's equation** (see Sec. 5.3), written out

(15′) $$(1 - w^2)\frac{d^2H}{dw^2} - 2w\frac{dH}{dw} + n(n+1)H = 0.$$

For integer $n = 0, 1, \cdots$ the Legendre polynomials

$$H = P_n(w) = P_n(\cos\phi) \qquad n = 0, 1, \cdots,$$

are solutions of Legendre's equation (15). We thus obtain the following two sequences of solution $u = GH$ of Laplace's equation (8), with constant A_n and B_n, where $n = 0, 1, \cdots,$

(16) $$\text{(a)}\quad u_n(r, \phi) = A_n r^n P_n(\cos\phi), \qquad \text{(b)}\quad u_n^*(r, \phi) = \frac{B_n}{r^{n+1}} P_n(\cos\phi)$$

Use of Fourier–Legendre Series

Interior Problem: Potential Within the Sphere S. We consider a series of terms from (16a),

(17) $$u(r, \phi) = \sum_{n=0}^{\infty} A_n r^n P_n(\cos\phi) \qquad (r \leqq R).$$

Since S is given by $r = R$, for (17) to satisfy the Dirichlet condition (9) on the sphere S, we must have

(18) $$u(R, \phi) = \sum_{n=0}^{\infty} A_n R^n P_n(\cos\phi) = f(\phi);$$

that is, (18) must be the **Fourier–Legendre series** of $f(\phi)$. From (7) in Sec. 5.8 we get the coefficients

(19*) $$A_n R^n = \frac{2n+1}{2}\int_{-1}^{1} \tilde{f}(w)\, P_n(w)\, dw$$

where $\tilde{f}(w)$ denotes $f(\phi)$ as a function of $w = \cos\phi$. Since $dw = -\sin\phi\, d\phi$, and the limits of integration -1 and 1 correspond to $\phi = \pi$ and $\phi = 0$, respectively, we also obtain

(19) $$A_n = \frac{2n+1}{2R^n}\int_0^{\pi} f(\phi) P_n(\cos\phi)\sin\phi\, d\phi, \qquad n = 0, 1, \cdots.$$

If $f(\phi)$ and $f'(\phi)$ are piecewise continuous on the interval $0 \leqq \phi \leqq \pi$, then the series (17) with coefficients (19) solves our problem for points inside the sphere because it can be shown that under these continuity assumptions the series (17) with coefficients (19) gives the derivatives occurring in (8) by termwise differentiation, thus justifying our derivation.

Exterior Problem: Potential Outside the Sphere S. Outside the sphere we cannot use the functions u_n in (16a) because they do not satisfy (10). But we can use the u_n^* in (16b), which do satisfy (10) (but could not be used inside S; why?). Proceeding as before leads to the solution of the exterior problem

$$u(r, \phi) = \sum_{n=0}^{\infty} \frac{B_n}{r^{n+1}} P_n(\cos \phi) \qquad (r \geqq R) \tag{20}$$

satisfying (8), (9), (10), with coefficients

$$B_n = \frac{2n+1}{2} R^{n+1} \int_0^{\pi} f(\phi) P_n(\cos \phi) \sin \phi \, d\phi. \tag{21}$$

The next example illustrates all this for a sphere of radius 1 consisting of two hemispheres that are separated by a small strip of insulating material along the equator, so that these hemispheres can be kept at different potentials (110 V and 0 V).

EXAMPLE 1 Spherical Capacitor

Find the potential inside and outside a spherical capacitor consisting of two metallic hemispheres of radius 1 ft separated by a small slit for reasons of insulation, if the upper hemisphere is kept at 110 V and the lower is grounded (Fig. 313).

Solution. The given boundary condition is (recall Fig. 312)

$$f(\phi) = \begin{cases} 110 & \text{if} \quad 0 \leqq \phi < \pi/2 \\ 0 & \text{if} \quad \pi/2 < \phi \leqq \pi. \end{cases}$$

Since $R = 1$, we thus obtain from (19)

$$A_n = \frac{2n+1}{2} \cdot 110 \int_0^{\pi/2} P_n(\cos \phi) \sin \phi \, d\phi$$

$$= \frac{2n+1}{2} \cdot 110 \int_0^{1} P_n(w) \, dw$$

where $w = \cos \phi$. Hence $P_n(\cos \phi) \sin \phi \, d\phi = -P_n(w) \, dw$, we integrate from 1 to 0, and we finally get rid of the minus by integrating from 0 to 1. You can evaluate this integral by your CAS or continue by using (11) in Sec. 5.2, obtaining

$$A_n = 55(2n+1) \sum_{m=0}^{M} (-1)^m \frac{(2n-2m)!}{2^n m!(n-m)!(n-2m)!} \int_0^1 w^{n-2m} \, dw$$

where $M = n/2$ for even n and $M = (n-1)/2$ for odd n. The integral equals $1/(n - 2m + 1)$. Thus

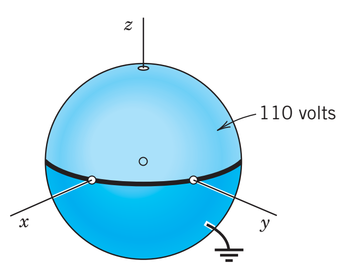

Fig. 313. Spherical capacitor in Example 1

$$A_n = \frac{55(2n+1)}{2^n} \sum_{m=0}^{M} (-1)^m \frac{(2n-2m)!}{m!(n-m)!(n-2m+1)!}. \tag{22}$$

Taking $n = 0$, we get $A_0 = 55$ (since $0! = 1$). For $n = 1, 2, 3, \cdots$ we get

$$A_1 = \frac{165}{2} \cdot \frac{2!}{0!1!2!} = \frac{165}{2},$$

$$A_2 = \frac{275}{4}\left(\frac{4!}{0!2!3!} - \frac{2!}{1!1!1!}\right) = 0,$$

$$A_3 = \frac{385}{8}\left(\frac{6!}{0!3!4!} - \frac{4!}{1!2!2!}\right) = -\frac{385}{8}, \qquad \text{etc.}$$

Hence the ***potential* (17) *inside the sphere*** is (since $P_0 = 1$)

$$u(r, \phi) = 55 + \frac{165}{2} r P_1(\cos\phi) - \frac{385}{8} r^3 P_3(\cos\phi) + \cdots \tag{23}$$ (Fig. 314)

with $P_1, P_3, \cdots$ given by (11′), Sec. 5.21. Since $R = 1$, we see from (19) and (21) in this section that $B_n = A_n$, and (20) thus gives the ***potential outside the sphere***

$$u(r, \phi) = \frac{55}{r} + \frac{165}{2r^2} P_1(\cos\phi) - \frac{385}{8r^4} P_3(\cos\phi) + \cdots. \tag{24}$$

Partial sums of these series can now be used for computing approximate values of the inner and outer potential. Also, it is interesting to see that far away from the sphere the potential is approximately that of a point charge, namely, $55/r$. (Compare with Theorem 3 in Sec. 9.7.) ■

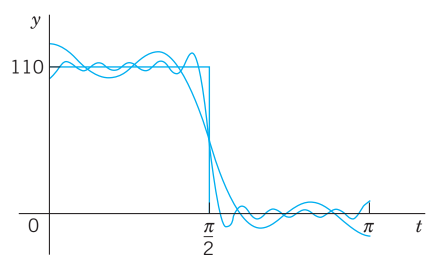

Fig. 314. Partial sums of the first 4, 6, and 11 nonzero terms of (23) for $r = R = 1$

EXAMPLE 2 Simpler Cases. Help with Problems

The technicalities encountered in cases that are similar to the one shown in Example 1 can often be avoided. For instance, find the potential inside the sphere S: $r = R = 1$ when S is kept at the potential $f(\phi) = \cos 2\phi$. (Can you see the potential on S? What is it at the North Pole? The equator? The South Pole?)

Solution. $w = \cos\phi$, $\cos 2\phi = 2\cos^2\phi - 1 = 2w^2 - 1 = \frac{4}{3}P_2(w) - \frac{1}{3} = \frac{4}{3}(\frac{3}{2}w^2 - \frac{1}{2}) - \frac{1}{3}$. Hence the potential in the interior of the sphere is

$$u = \frac{4}{3}r^2 P_2(w) - \frac{1}{3} = \frac{4}{3}r^2 P_2(\cos\phi) - \frac{1}{3} = \frac{2}{3}r^2(3\cos^2\phi - 1) - \frac{1}{3}.$$ ■

PROBLEM SET 12.11

1. **Spherical coordinates.** Derive (7) from $\nabla^2 u$ in spherical coordinates.
2. **Cylindrical coordinates.** Verify (5) by transforming $\nabla^2 u$ back into Cartesian coordinates.
3. Sketch $P_n(\cos\theta)$, $0 \leqq \theta \leqq 2\pi$, for $n = 0, 1, 2$. (Use (11′) in Sec. 5.2.)
4. **Zero surfaces.** Find the surfaces on which u_1, u_2, u_3 in (16) are **zero**.

5. CAS PROBLEM. Partial Sums. In Example 1 in the text verify the values of A_0, A_1, A_2, A_3 and compute $A_4, \cdots, A_{10}$. Try to find out graphically how well the corresponding partial sums of (23) approximate the given boundary function.

6. CAS EXPERIMENT. Gibbs Phenomenon. Study the Gibbs phenomenon in Example 1 (Fig. 314) graphically.

7. Verify that u_n and u_n^* in (16) are solutions of (8).

8–15 POTENTIALS DEPENDING ONLY ON r

8. Dimension 3. Verify that the potential $u = c/r$, $r = \sqrt{x^2 + y^2 + z^2}$ satisfies Laplace's equation in spherical coordinates.

9. Spherical symmetry. Show that the only solution of Laplace's equation depending only on $r = \sqrt{x^2 + y^2 + z^2}$ is $u = c/r + k$ with constant c and k.

10. Cylindrical symmetry. Show that the only solution of Laplace's equation depending only on $r = \sqrt{x^2 + y^2}$ is $u = c \ln r + k$.

11. Verification. Substituting $u(r)$ with r as in Prob. 9 into $u_{xx} + u_{yy} + u_{zz} = 0$, verify that $u'' + 2u'/r = 0$, in agreement with (7).

12. Dirichlet problem. Find the electrostatic potential between coaxial cylinders of radii $r_1 = 2$ cm and $r_2 = 4$ cm kept at the potentials $U_1 = 220$ V and $U_2 = 140$ V, respectively.

13. Dirichlet problem. Find the electrostatic potential between two concentric spheres of radii $r_1 = 2$ cm and $r_2 = 4$ cm kept at the potentials $U_1 = 220$ V and $U_2 = 140$ V, respectively. Sketch and compare the equipotential lines in Probs. 12 and 13. Comment.

14. Heat problem. If the surface of the ball $r^2 = x^2 + y^2 + z^2 \leqq R^2$ is kept at temperature zero and the initial temperature in the ball is $f(r)$, show that the temperature $u(r, t)$ in the ball is a solution of $u_t = c^2(u_{rr} + 2u_r/r)$ satisfying the conditions $u(R, t) = 0$, $u(r, 0) = f(r)$. Show that setting $v = ru$ gives $v_t = c^2 v_{rr}$, $v(R, t) = 0$, $v(r, 0) = rf(r)$. Include the condition $v(0, t) = 0$ (which holds because u must be bounded at $r = 0$), and solve the resulting problem by separating variables.

15. What are the analogs of Probs. 12 and 13 in heat conduction?

16–20 BOUNDARY VALUE PROBLEMS IN SPHERICAL COORDINATES r, θ, ϕ

Find the potential in the interior of the sphere $r = R = 1$ if the interior is free of charges and the potential on the sphere is

16. $f(\phi) = \cos \phi$

17. $f(\phi) = 1$

18. $f(\phi) = 1 - \cos^2 \phi$

19. $f(\phi) = \cos 2\phi$

20. $f(\phi) = 10 \cos^3 \phi - 3 \cos^2 \phi - 5 \cos \phi - 1$

21. Point charge. Show that in Prob. 17 the potential exterior to the sphere is the same as that of a point charge at the origin.

22. Exterior potential. Find the potentials exterior to the sphere in Probs. 16 and 19.

23. Plane intersections. Sketch the intersections of the equipotential surfaces in Prob. 16 with xz-plane.

24. TEAM PROJECT. Transmission Line and Related PDEs. Consider a long cable or telephone wire (Fig. 315) that is imperfectly insulated, so that leaks occur along the entire length of the cable. The source S of the current $i(x, t)$ in the cable is at $x = 0$, the receiving end T at $x = l$. The current flows from S to T and through the load, and returns to the ground. Let the constants R, L, C, and G denote the resistance, inductance, capacitance to ground, and conductance to ground, respectively, of the cable per unit length.

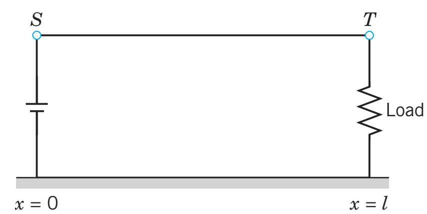

Fig. 315. Transmission line

(a) Show that ("**first transmission line equation**")

$$-\frac{\partial u}{\partial x} = Ri + L\frac{\partial i}{\partial t}$$

where $u(x, t)$ is the potential in the cable. *Hint:* Apply Kirchhoff's voltage law to a small portion of the cable between x and $x + \Delta x$ (difference of the potentials at x and $x + \Delta x$ = resistive drop + inductive drop).

(b) Show that for the cable in (a) ("**second transmission line equation**"),

$$-\frac{\partial i}{\partial x} = Gu + C\frac{\partial u}{\partial t}.$$

Hint: Use Kirchhoff's current law (difference of the currents at x and $x + \Delta x$ = loss due to leakage to ground + capacitive loss).

(c) Second-order PDEs. Show that elimination of i or u from the transmission line equations leads to

$$u_{xx} = LCu_{tt} + (RC + GL)u_t + RGu,$$
$$i_{xx} = LCi_{tt} + (RC + GL)i_t + RGi.$$

(d) Telegraph equations. For a submarine cable, G is negligible and the frequencies are low. Show that this leads to the so-called *submarine cable equations* or **telegraph equations**

$$u_{xx} = RCu_t, \qquad i_{xx} = RCi_t.$$

Find the potential in a submarine cable with ends ($x = 0, x = l$) grounded and initial voltage distribution $U_0 = \text{const}$.

(e) High-frequency line equations. Show that in the case of alternating currents of high frequencies the equations in (c) can be approximated by the so-called **high-frequency line equations**

$$u_{xx} = LCu_{tt}, \qquad i_{xx} = LCi_{tt}.$$

Solve the first of them, assuming that the initial potential is

$$U_0 \sin(\pi x/l),$$

and $u_t(x, 0) = 0$ and $u = 0$ at the ends $x = 0$ and $x = l$ for all t.

25. Reflection in a sphere. Let r, θ, ϕ be spherical coordinates. If $u(r, \theta, \phi)$ satisfies $\nabla^2 u = 0$, show that $v(r, \theta, \phi) = u(1/r, \theta, \phi)/r$ satisfies $\nabla^2 v = 0$.

12.12 Solution of PDEs by Laplace Transforms

Readers familiar with Chap. 6 may wonder whether Laplace transforms can also be used for solving *partial* differential equations. The answer is yes, particularly if one of the independent variables ranges over the positive axis. The steps to obtain a solution are similar to those in Chap. 6. For a PDE in two variables they are as follows.

1. Take the Laplace transform with respect to one of the two variables, usually t. This gives an ***ODE for the transform*** of the unknown function. This is so since the derivatives of this function with respect to the other variable slip into the transformed equation. The latter also incorporates the given boundary and initial conditions.
2. Solving that ODE, obtain the transform of the unknown function.
3. Taking the inverse transform, obtain the solution of the given problem.

If the coefficients of the given equation do not depend on t, the use of Laplace transforms will simplify the problem.

We explain the method in terms of a typical example.

EXAMPLE 1 Semi-Infinite String

Find the displacement $w(x, t)$ of an elastic string subject to the following conditions. (We write w since we need u to denote the unit step function.)

(i) The string is initially at rest on the x-axis from $x = 0$ to ∞ (*"semi-infinite string"*).

(ii) For $t > 0$ the left end of the string ($x = 0$) is moved in a given fashion, namely, according to a single sine wave

$$w(0, t) = f(t) = \begin{cases} \sin t & \text{if } 0 \leqq t \leqq 2\pi \\ 0 & \text{otherwise} \end{cases} \qquad \text{(Fig. 316).}$$

(iii) Furthermore, $\lim_{x\to\infty} w(x, t) = 0$ for $t \geqq 0$.

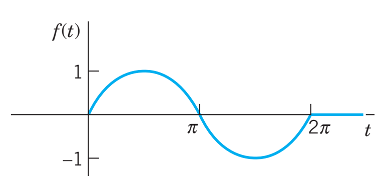

Fig. 316. Motion of the left end of the string in Example 1 as a function of time t