Laplace’s Equation in Cylindrical and Spherical Coordinates. Potential

One of the most important PDEs in physics and engineering applications is Laplace’s equation, given by

\[ \nabla^2 u = u_{xx} + u_{yy} + u_{zz} = 0. \tag{1} \]

Here, \( x, y, z \) are Cartesian coordinates in space (Fig. 167 in Sec. 9.1), \( u_{xx} = \partial^2 u / \partial x^2 \), etc. The expression \( \nabla^2 u \) is called the Laplacian of \( u \). The theory of the solutions of (1) is called potential theory. Solutions of (1) that have continuous second partial derivatives are known as harmonic functions.

Laplace’s equation occurs mainly in gravitation, electrostatics (see Theorem 3, Sec. 9.7), steady-state heat flow (Sec. 12.5), and fluid flow (to be discussed in Sec. 18.4).

Recall from Sec. 9.7 that the gravitational potential \( u(x, y, z) \) at a point \( (x, y, z) \) resulting from a single mass located at a point \( (X, Y, Z) \) is

\[ u(x, y, z) = \frac{c}{r} = \frac{c}{\sqrt{(x - X)^2 + (y - Y)^2 + (z - Z)^2}} \quad (r > 0) \tag{2} \]

and \( u \) satisfies (1). Similarly, if mass is distributed in a region \( T \) in space with density \( \rho(X, Y, Z) \), its potential at a point \( (x, y, z) \) not occupied by mass is

\[ u(x, y, z) = k \iiint_T \frac{\rho(X, Y, Z)}{r} \, dX dY dZ. \tag{3} \]

It satisfies (1) because \( \nabla^2 (1/r) = 0 \) (Sec. 9.7) and \( \rho \) is not a function of \( x, y, z \).

Practical problems involving Laplace’s equation are boundary value problems in a region \( T \) in space with boundary surface \( S \). Such problems can be grouped into three types (see also Sec. 12.6 for the two-dimensional case):

(I) First boundary value problem or Dirichlet problem if \( u \) is prescribed on \( S \).

(II) Second boundary value problem or Neumann problem if the normal derivative \( u_n = \partial u / \partial n \) is prescribed on \( S \).

(III) Third or mixed boundary value problem or Robin problem if \( u \) is prescribed on a portion of \( S \) and \( u_n \) on the remaining portion of \( S \).

In general, when we want to solve a boundary value problem, we have to first select the appropriate coordinates in which the boundary surface \( S \) has a simple representation. Here are some examples followed by some applications.

Laplacian in Cylindrical Coordinates

The first step in solving a boundary value problem is generally the introduction of coordinates in which the boundary surface \( S \) has a simple representation. Cylindrical symmetry (a cylinder as a region \( T \)) calls for cylindrical coordinates \( r, \theta, z \) related to \( x, y, z \) by

\[ x = r \cos \theta, \quad y = r \sin \theta, \quad z = z \tag{4} \quad \text{(Fig. 311)}. \]
For these we get $\nabla^2 u$ immediately by adding $u_{zz}$ to (5) in Sec. 12.10; thus,

\[ \nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2}. \]

**Laplacian in Spherical Coordinates**

Spherical symmetry (a ball as region $T$ bounded by a sphere $S$) requires **spherical coordinates** $r, \theta, \phi$ related to $x, y, z$ by

\[ x = r \cos \theta \sin \phi, \quad y = r \sin \theta \sin \phi, \quad z = r \cos \phi \quad \text{(Fig. 312)}. \]

Using the chain rule (as in Sec. 12.10), we obtain $\nabla^2 u$ in spherical coordinates

\[ \nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{2}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \phi^2} + \frac{\cot \phi}{r} \frac{\partial u}{\partial \phi} + \frac{1}{r^2 \sin^2 \phi} \frac{\partial^2 u}{\partial \theta^2}. \]

We leave the details as an exercise. It is sometimes practical to write (7) in the form

\[ \nabla^2 u = \frac{1}{r^2} \left[ \frac{\partial}{\partial r} \left( r^2 \frac{\partial u}{\partial r} \right) + \frac{1}{\sin \phi} \frac{\partial}{\partial \phi} \left( \sin \phi \frac{\partial u}{\partial \phi} \right) + \frac{1}{\sin^2 \phi} \frac{\partial^2 u}{\partial \theta^2} \right]. \]

**Remark on Notation.** Equation (6) is used in calculus and extends the familiar notation for polar coordinates. Unfortunately, some books use $\theta$ and $\phi$ interchanged, an extension of the notation $x = r \cos \phi, y = r \sin \phi$ for polar coordinates (used in some European countries).

**Boundary Value Problem in Spherical Coordinates**

We shall solve the following **Dirichlet problem** in spherical coordinates:

\[ \nabla^2 u = \frac{1}{r^2} \left[ \frac{\partial}{\partial r} \left( r^2 \frac{\partial u}{\partial r} \right) + \frac{1}{\sin \phi} \frac{\partial}{\partial \phi} \left( \sin \phi \frac{\partial u}{\partial \phi} \right) \right] = 0. \]

\[ u(R, \phi) = f(\phi) \]

\[ \lim_{r \to 0} u(r, \phi) = 0. \]
The PDE (8) follows from (7) or by assuming that the solution \( u \) will not depend on \( \theta \) because the Dirichlet condition (9) is independent of \( \theta \). This may be an electrostatic potential (or a temperature) \( f(\phi) \) at which the sphere \( S: r = R \) is kept. Condition (10) means that the potential at infinity will be zero.

**Separating Variables** by substituting \( u(r, \phi) = G(r)H(\phi) \) into (8). Multiplying (8) by \( r^2 \), making the substitution and then dividing by \( GH \), we obtain

\[
\frac{1}{G} \frac{d}{dr} \left( r^2 \frac{dG}{dr} \right) = - \frac{1}{H \sin \phi} \frac{d}{d\phi} \left( \sin \phi \frac{dH}{d\phi} \right).
\]

By the usual argument both sides must be equal to a constant \( k \). Thus we get the two ODEs

\[
\frac{1}{G} \frac{d}{dr} \left( r^2 \frac{dG}{dr} \right) = k \quad \text{or} \quad r^2 \frac{d^2G}{dr^2} + 2r \frac{dG}{dr} = kG
\]

and

\[
\frac{1}{\sin \phi} \frac{d}{d\phi} \left( \sin \phi \frac{dH}{d\phi} \right) + kH = 0.
\]

The solutions of (11) will take a simple form if we set \( k = n(n + 1) \). Then, writing \( G' = dG/dr \), etc., we obtain

\[
r^2G'' + 2rG' - n(n + 1) G = 0.
\]

This is an Euler–Cauchy equation. From Sec. 2.5 we know that it has solutions \( G = r^a \). Substituting this and dropping the common factor \( r^a \) gives

\[
a(a - 1) + 2a - n(n + 1) = 0. \quad \text{The roots are} \quad a = n \quad \text{and} \quad -n - 1.
\]

Hence solutions are

\[
G_n(r) = r^n \quad \text{and} \quad G_n^*(r) = \frac{1}{r^{n+1}}.
\]

We now solve (12). Setting \( \cos \phi = w \), we have \( \sin^2 \phi = 1 - w^2 \) and

\[
\frac{d}{d\phi} = \frac{d}{dw} \frac{dw}{d\phi} = -\sin \phi \frac{d}{dw}
\]

Consequently, (12) with \( k = n(n + 1) \) takes the form

\[
\frac{d}{dw} \left[ (1 - w^2) \frac{dH}{dw} \right] + n(n + 1)H = 0.
\]

This is Legendre’s equation (see Sec. 5.3), written out
For integer \( n = 0, 1, \cdots \) the Legendre polynomials
\[
H = P_n(w) = P_n(\cos \phi)
\]
are solutions of Legendre’s equation (15). We thus obtain the following two sequences of solution of Laplace’s equation (8), with constant \( A_n \) and \( B_n \), where \( n = 0, 1, \cdots \),
\[
(16)
\]
\[
(a) \quad u_n(r, \phi) = A_n r^n P_n(\cos \phi), \quad (b) \quad u_n^*(r, \phi) = \frac{B_n}{r^{n+1}} P_n(\cos \phi)
\]

Use of Fourier–Legendre Series

Interior Problem: Potential Within the Sphere \( S \). We consider a series of terms from (16a),
\[
(17)
\]
\[
u(r, \phi) = \sum_{n=0}^{\infty} A_n r^n P_n(\cos \phi) \quad (r \leq R).
\]

Since \( S \) is given by \( r = R \), for (17) to satisfy the Dirichlet condition (9) on the sphere \( S \), we must have
\[
(18)
\]
\[
u(R, \phi) = \sum_{n=0}^{\infty} A_n R^n P_n(\cos \phi) = f(\phi);
\]
that is, (18) must be the Fourier–Legendre series of \( f(\phi) \). From (7) in Sec. 5.8 we get the coefficients
\[
(19^*)
\]
\[
A_n R^n = \frac{2n + 1}{2} \int_{-1}^{1} \tilde{f}(w) P_n(w) \, dw
\]
where \( \tilde{f}(w) \) denotes \( f(\phi) \) as a function of \( w = \cos \phi \). Since \( dw = -\sin \phi \, d\phi \), and the limits of integration \(-1 \) and \( 1 \) correspond to \( \phi = \pi \) and \( \phi = 0 \), respectively, we also obtain
\[
(19)
\]
\[
A_n = \frac{2n + 1}{2R^n} \int_{0}^{\pi} f(\phi)P_n(\cos \phi) \sin \phi \, d\phi, \quad n = 0, 1, \cdots
\]

If \( f(\phi) \) and \( f'(\phi) \) are piecewise continuous on the interval \( 0 \leq \phi \leq \pi \), then the series (17) with coefficients (19) solves our problem for points inside the sphere because it can be shown that under these continuity assumptions the series (17) with coefficients (19) gives the derivatives occurring in (8) by termwise differentiation, thus justifying our derivation.
Exterior Problem: Potential Outside the Sphere $S$. Outside the sphere we cannot use the functions $u_n$ in (16a) because they do not satisfy (10). But we can use the in (16b), which do satisfy (10) (but could not be used inside $S$; why?). Proceeding as before leads to the solution of the exterior problem

$$u(r, \phi) = \sum_{n=0}^{\infty} \frac{B_n}{r^{n+1}} P_n(\cos \phi) \quad (r \geq R)$$

satisfying (8), (9), (10), with coefficients

$$B_n = \frac{2n + 1}{2} R^{n+1} \int_0^{\pi/2} f(\phi) P_n(\cos \phi) \sin \phi \, d\phi.$$ 

The next example illustrates all this for a sphere of radius 1 consisting of two hemispheres that are separated by a small strip of insulating material along the equator, so that these hemispheres can be kept at different potentials (110 V and 0 V).

**Example 1**: Spherical Capacitor

Find the potential inside and outside a spherical capacitor consisting of two metallic hemispheres of radius 1 ft separated by a small slit for reasons of insulation, if the upper hemisphere is kept at 110 V and the lower is grounded (Fig. 313).

**Solution.** The given boundary condition is (recall Fig. 312)

$$f(\phi) = \begin{cases} 110 & \text{if } 0 \leq \phi < \pi/2 \\ 0 & \text{if } \pi/2 < \phi \leq \pi. \end{cases}$$

Since $R = 1$, we thus obtain from (19)

$$A_n = \frac{2n + 1}{2} \cdot 110 \int_0^{\pi/2} P_n(\cos \phi) \sin \phi \, d\phi = \frac{2n + 1}{2} \cdot 110 \int_0^{1} P_n(w) \, dw,$$

where $w = \cos \phi$. Hence $P_n(\cos \phi) \sin \phi \, d\phi = -P_n(w) \, dw$, we integrate from 1 to 0, and we finally get rid of the minus by integrating from 0 to 1. You can evaluate this integral by your CAS or continue by using (11) in Sec. 5.2, obtaining

$$A_n = 55(2n + 1) \sum_{m=0}^{M} (-1)^m \frac{(2n - 2m)!}{2^m m!(n - m)!(n - 2m)!} \int_0^1 w^{n-2m} \, dw,$$

where $M = n/2$ for even $n$ and $M = (n - 1)/2$ for odd $n$. The integral equals $1/(n - 2m + 1)$. Thus
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(22) \[ A_n = \frac{55(2n + 1)}{2^n} \sum_{m=0}^{n} \frac{(-1)^m (2n - 2m)!}{m!(n - m)!(n - 2m + 1)!}. \]

Taking \( n = 0 \), we get \( A_0 = 55 \) (since \( 0! = 1 \)). For \( n = 1, 2, 3, \ldots \) we get

\[ A_1 = \frac{165}{2}, \quad A_2 = 275/4, \quad A_3 = 385/8, \]
\[ A_4 = 275/4, \quad A_5 = 385/8, \quad \text{etc.} \]

Hence the potential (17) inside the sphere is (since \( P_0 = 1 \))

(23) \[ u(r, \phi) = 55 + \frac{165}{2} r P_1(\cos \phi) - \frac{385}{8} r^2 P_2(\cos \phi) + \cdots \]

with \( P_1, P_2, \ldots \) given by (11’), Sec. 5.21. Since \( R = 1 \), we see from (19) and (21) in this section that \( B_n = A_n \), and (20) thus gives the potential outside the sphere

(24) \[ u(r, \phi) = \frac{55}{r} + \frac{165}{2r^2} P_1(\cos \phi) - \frac{385}{8r^4} P_2(\cos \phi) + \cdots. \]

Partial sums of these series can now be used for computing approximate values of the inner and outer potential. Also, it is interesting to see that far away from the sphere the potential is approximately that of a point charge, namely, \( 55/r \). (Compare with Theorem 3 in Sec. 9.7.)

**Example 2** Simpler Cases. Help with Problems

The technicalities encountered in cases that are similar to the one shown in Example 1 can often be avoided. For instance, find the potential inside the sphere \( S: r = R = 1 \) when \( S \) is kept at the potential \( f(\phi) = \cos 2\phi \).

*Can you see the potential on \( S \)? What is it at the North Pole? The equator? The South Pole?*

**Solution.** \( w = \cos \phi, \cos 2\phi = 2\cos^2 \phi - 1 = 2w^2 - 1 = \frac{2}{3} P_2(w) - \frac{1}{3} = \frac{2}{3} (2w^2 - 1) - \frac{1}{3} \). Hence the potential in the interior of the sphere is

\[ u = \frac{2}{3} r^2 P_2(w) - \frac{1}{3} = \frac{2}{3} r^2 P_2(\cos \phi) - \frac{1}{3} = \frac{2}{3} r^2 (3 \cos^2 \phi - 1) - \frac{1}{3}. \]

**Problem Set 12.11**

1. **Spherical coordinates.** Derive (7) from \( \nabla^2 u \) in spherical coordinates.
2. **Cylindrical coordinates.** Verify (5) by transforming \( \nabla^2 u \) back into Cartesian coordinates.
3. Sketch \( P_n(\cos \theta), 0 \leq \theta \leq 2\pi \), for \( n = 0, 1, 2, \) (Use (11’) in Sec. 5.2.)
4. **Zero surfaces.** Find the surfaces on which \( u_1, u_2, u_3 \) in (16) are zero.
5. **CAS PROBLEM. Partial Sums.** In Example 1 in the text verify the values of \( A_0, A_1, A_2, A_3 \) and compute \( A_4, \ldots, A_{10} \). Try to find out graphically how well the corresponding partial sums of (23) approximate the given boundary function.

6. **CAS EXPERIMENT. Gibbs Phenomenon.** Study the Gibbs phenomenon in Example 1 (Fig. 314) graphically.

7. Verify that \( u_n \) and \( u_{n0}^2 \) in (16) are solutions of (8).

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**POTENTIALS DEPENDING ONLY ON** \( r \)

8. **Dimension 3.** Verify that the potential \( u = c/r, r = \sqrt{x^2 + y^2 + z^2} \) satisfies Laplace’s equation in spherical coordinates.

9. **Spherical symmetry.** Show that the only solution of Laplace’s equation depending only on \( r = \sqrt{x^2 + y^2 + z^2} \) is \( u = c/r + k \) with constant \( c \) and \( k \).

10. **Cylindrical symmetry.** Show that the only solution of Laplace’s equation depending only on \( r = \sqrt{x^2 + y^2} \) is \( u = c \ln r + k \).

11. **Verification.** Substituting \( u(r) \) with \( r \) as in Prob. 9 into \( u_{xx} + u_{yy} + u_{zz} = 0 \), verify that \( u'' + 2u'/r = 0 \), in agreement with (7).

12. **Dirichlet problem.** Find the electrostatic potential between coaxial cylinders of radii \( r_1 = 2 \) cm and \( r_2 = 4 \) cm kept at the potentials \( U_1 = 220 \) V and \( U_2 = 140 \) V, respectively.

13. **Dirichlet problem.** Find the electrostatic potential between two concentric spheres of radii \( r_1 = 2 \) cm and \( r_2 = 4 \) cm kept at the potentials \( U_1 = 220 \) V and \( U_2 = 140 \) V, respectively. Sketch and compare the equipotential lines in Probs. 12 and 13. Comment.

14. **Heat problem.** If the surface of the ball \( r^2 = x^2 + y^2 + z^2 \leq R^2 \) is kept at temperature zero and the initial temperature in the ball is \( f(r) \), show that the temperature \( u(r, t) \) in the ball is a solution of \( u_t = c^2 (u_{rr} + 2u_r/r) \) satisfying the conditions \( u(R, t) = 0, u(r, 0) = f(r) \). Show that setting \( v = ru \) gives \( v_t = c^2 v_{rr}, v(R, t) = 0, v(r, 0) = rf(r) \). Include the condition \( v(0, t) = 0 \) (which holds because \( u \) must be bounded at \( r = 0 \)), and solve the resulting problem by separating variables.

15. What are the analogs of Probs. 12 and 13 in heat conduction?

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**BOUNDARY VALUE PROBLEMS IN SPHERICAL COORDINATES** \( r, \theta, \phi \)

Find the potential in the interior of the sphere \( r = R = 1 \) if the interior is free of charges and the potential on the sphere is

16. \( f(\phi) = \cos \phi \)  
17. \( f(\phi) = 1 \)
18. \( f(\phi) = 1 - \cos^2 \phi \)  
19. \( f(\phi) = \cos 2\phi \)
20. \( f(\phi) = 10 \cos^3 \phi - 3 \cos^2 \phi - 5 \cos \phi - 1 \)

21. **Point charge.** Show that in Prob. 17 the potential exterior to the sphere is the same as that of a point charge at the origin.

22. **Exterior potential.** Find the potentials exterior to the sphere in Probs. 16 and 19.

23. **Plane intersections.** Sketch the intersections of the equipotential lines in Prob. 16 with \( xz \)-plane.

24. **TEAM PROJECT. Transmission Line and Related PDEs.** Consider a long cable or telephone wire (Fig. 315) that is imperfectly insulated, so that leaks occur along the entire length of the cable. The source \( S \) of the current \( i(x, t) \) in the cable is at \( x = 0 \), the receiving end \( T \) at \( x = l \). The current flows from \( S \) to \( T \) through the load, and returns to the ground. Let the constants \( R, L, C, \) and \( G \) denote the resistance, inductance, capacitance to ground, and conductance to ground, respectively, of the cable per unit length.

(a) Show that (“first transmission line equation”)

\[ \frac{\partial u}{\partial x} = Ri + L \frac{\partial i}{\partial t} \]

where \( u(x, t) \) is the potential in the cable. Hint: Apply Kirchhoff’s voltage law to a small portion of the cable between \( x \) and \( x + \Delta x \) (difference of the potentials at \( x \) and \( x + \Delta x \) = resistive drop + inductive drop).

(b) Show that for the cable in (a) (“second transmission line equation”),

\[ \frac{\partial i}{\partial x} = Gu + C \frac{\partial u}{\partial t}. \]

Hint: Use Kirchhoff’s current law (difference of the currents at \( x \) and \( x + \Delta x \) = loss due to leakage to ground + capacitive loss).

(c) **Second-order PDEs.** Show that elimination of \( i \) or \( u \) from the transmission line equations leads to

\[ u_{xx} = LCU_{tt} + (RC + GL)u_t + RGu, \]
\[ i_{xx} = LCI_{tt} + (RC + GL)i_t + CGI. \]

(d) **Telegraph equations.** For a submarine cable, \( G \) is negligible and the frequencies are low. Show that this leads to the so-called submarine cable equations or telegraph equations

\[ u_{xx} = RCu_t, \quad i_{xx} = RCi_t. \]
Find the potential in a submarine cable with ends \((x = 0, x = l)\) grounded and initial voltage distribution \(U_0 = \text{const.}\).

**Example 1 Semi-Infinite String**

Find the displacement \(w(x, t)\) of an elastic string subject to the following conditions. (We write \(w\) since we need \(u\) to denote the unit step function.)

1. The string is initially at rest on the \(x\)-axis from \(x = 0\) to \(\infty\) ("semi-infinite string").
2. For \(t > 0\) the left end of the string \((x = 0)\) is moved in a given fashion, namely, according to a single sine wave

\[
w(0, t) = f(t) = \begin{cases} \sin t & \text{if } 0 \leq t \leq 2\pi \\ 0 & \text{otherwise} \end{cases}
\]

(Fig. 316).

3. Furthermore, \(\lim_{x \to \infty} w(x, t) = 0\) for \(t \geq 0\).

Solve the first of them, assuming that the initial potential is

\[U_0 \sin \left(\pi x / l\right),\]

and \(u_t(x, 0) = 0\) and \(u = 0\) at the ends \(x = 0\) and \(x = l\) for all \(t\).

25. Reflection in a sphere. Let \(r, \theta, \phi\) be spherical coordinates. If \(u(r, \theta, \phi)\) satisfies \(\nabla^2 u = 0\), show that \(v(r, \theta, \phi) = u(1/r, \theta, \phi)/r\) satisfies \(\nabla^2 v = 0\).