MATH 401 - NOTES

Sequences of functions
Pointwise and Uniform Convergence

Fall 2005

Previously, we have studied sequences of *real numbers*. Now we discuss the topic of *sequences of real valued functions*. A sequence of functions $\{f_n\}$ is a list of functions $(f_1, f_2, \ldots)$ such that each f_n maps a given subset D of $\mathbb{R}$ into $\mathbb{R}$.

I. Pointwise convergence

Definition. *Let D be a subset of $\mathbb{R}$ and let $\{f_n\}$ be a sequence of functions defined on D. We say that $\{f_n\}$ converges pointwise on D if*

$$\lim_{n\to\infty} f_n(x) \textbf{ exists for each point } x \text{ in } D.$$

This means that $\lim_{n\to\infty} f_n(x)$ is a real number that depends only on x.

If $\{f_n\}$ is pointwise convergent then the function defined by $f(x) = \lim_{n\to\infty} f_n(x)$, for every x in D, is called the *pointwise limit of the sequence* $\{f_n\}$.

Example 1. Let $\{f_n\}$ be the sequence of functions on $\mathbb{R}$ defined by $f_n(x) = nx$. This sequence does not converge pointwise on $\mathbb{R}$ because $\lim_{n\to\infty} f_n(x) = \infty$ for any $x > 0$.

Example 2.
Let $\{f_n\}$ be the sequence of functions on $\mathbb{R}$ defined by $f_n(x) = x/n$. This sequence converges pointwise to the zero function on $\mathbb{R}$.

Example 3. *Consider the sequence $\{f_n\}$ of functions defined by*

$$f_n(x) = \frac{nx + x^2}{n^2} \qquad \text{for all } x \text{ in } \mathbb{R}.$$

Show that $\{f_n\}$ converges pointwise.

Solution: For every real number x, we have:

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{x}{n} + \frac{x^2}{n^2} = x\left(\lim_{n\to\infty} \frac{1}{n}\right) + x^2\left(\lim_{n\to\infty} \frac{1}{n^2}\right) = 0 + 0 = 0$$

Thus, $\{f_n\}$ converges pointwise to the zero function on $\mathbb{R}$.

Example 4. *Consider the sequence $\{f_n\}$ of functions defined by*

$$f_n(x) = \frac{\sin(nx+3)}{\sqrt{n+1}} \qquad \text{for all } x \text{ in } \mathbb{R}.$$

Show that $\{f_n\}$ converges pointwise.

Solution: For every x in $\mathbb{R}$, we have

$$\frac{-1}{\sqrt{n+1}} \leq \frac{\sin(nx+3)}{\sqrt{n+1}} \leq \frac{1}{\sqrt{n+1}}$$

Moreover,

$$\lim_{n\to\infty} \frac{1}{\sqrt{n+1}} = 0.$$

Applying the squeeze theorem for sequences, we obtain that

$$\lim_{n\to\infty} f_n(x) = 0 \qquad \text{for all } x \text{ in } \mathbb{R}.$$

Therefore, $\{f_n\}$ converges pointwise to the function $f \equiv 0$ on $\mathbb{R}$.

Example 5. *Consider the sequence $\{f_n\}$ of functions defined by $f_n(x) = n^2x^n$ for $0 \leq x \leq 1$. Determine whether $\{f_n\}$ is pointwise convergent.*

Solution: First of all, observe that $f_n(0) = 0$ for every n in $\mathbb{N}$. So the sequence $\{f_n(0)\}$ is constant and converges to zero. Now suppose $0 < x < 1$ then $n^2x^n = n^2e^{n\ln(x)}$. But $\ln(x) < 0$ when $0 < x < 1$, it follows that

$$\lim_{n\to\infty} f_n(x) = 0 \quad \text{for } 0 < x < 1$$

Finally, $f_n(1) = n^2$ for all n. So, $\lim\limits_{n\to\infty} f_n(1) = \infty$. Therefore, $\{f_n\}$ is not pointwise convergent on $[0, 1]$.

Example 6. *Let $\{f_n\}$ be the sequence of functions defined by $f_n(x) = \cos^n(x)$ for $-\pi/2 \leq x \leq \pi/2$. Discuss the pointwise convergence of the sequence.*

Solution: For $-\pi/2 \leq x < 0$ and for $0 < x \leq \pi/2$, we have

$$0 \leq \cos(x) < 1.$$

It follows that

$$\lim_{n\to\infty}(\cos(x))^n = 0 \qquad \text{for } x \neq 0.$$

Moreover, since $f_n(0) = 1$ for all n in $\mathbb{N}$, one gets $\lim_{n\to\infty} f_n(0) = 1$. Therefore, $\{f_n\}$ converges pointwise to the function f defined by

$$f(x) = \begin{cases} 0 & \text{if} \quad -\frac{\pi}{2} \leq x < 0 \quad \text{or} \quad 0 < x \leq \frac{\pi}{2} \\ 1 & \text{if} \qquad x = 0 \end{cases}$$

Example 7. *Consider the sequence of functions defined by*

$$f_n(x) = nx(1-x)^n \quad \text{on } [0,1].$$

Show that $\{f_n\}$ converges pointwise to the zero function.

Solution: Note that $f_n(0) = f_n(1) = 0$, for all $n \in \mathbb{N}$. Now suppose $0 < x < 1$, then

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} nxe^{n\ln(1-x)} = x \lim_{n\to\infty} ne^{n\ln(1-x)} = 0$$

because $\ln(1-x) < 0$ when $0 < x < 1$. Therefore, the given sequence converges pointwise to zero.

Example 8. *Let $\{f_n\}$ be the sequence of functions on $\mathbb{R}$ defined by*

$$f_n(x) = \begin{cases} n^3 & \text{if} \quad 0 < x \leq \frac{1}{n} \\ 1 & \text{otherwise} \end{cases}$$

Show that $\{f_n\}$ converges pointwise to the constant function $f = 1$ on $\mathbb{R}$.

Solution: For any x in $\mathbb{R}$ there is a natural number N such that x does not belong to the interval $(0,\ 1/N)$. The intervals $(0,\ 1/n)$ get smaller as $n \to \infty$. Therefore, $f_n(x) = 1$ for all $n > N$. Hence,

$$\lim_{n\to\infty} f_n(x) = 1 \quad \text{for all } x.$$

The formal definition of pointwise convergence

Let D be a subset of $\mathbb{R}$ and let $\{f_n\}$ be a sequence of real valued functions defined on D. Then $\{f_n\}$ converges pointwise to f if given any x in D and given any $\varepsilon > 0$, there exists a natural number $N = N(x, \varepsilon)$ such that

$$|f_n(x) - f(x)| < \varepsilon \quad \text{for every } n > N.$$

Note: The notation $N = N(x, \varepsilon)$ means that the natural number N depends on the choice of x and ε.

I. Uniform convergence

Definition. Let D be a subset of $\mathbb{R}$ and let $\{f_n\}$ be a sequence of real valued functions defined on D. Then $\{f_n\}$ converges uniformly to f if given any $\varepsilon > 0$, there exists a natural number $N = N(\varepsilon)$ such that

$$|f_n(x) - f(x)| < \varepsilon \quad \text{for every } n > N \quad \text{and for every } x \text{ in } D.$$

Note: In the above definition the natural number N depends only on ε. Therefore, uniform convergence implies pointwise convergence. But the converse is false as we can see from the following counter-example.

Example 9. *Let $\{f_n\}$ be the sequence of functions on $(0,\ \infty)$ defined by*

$$f_n(x) = \frac{nx}{1 + n^2x^2}.$$

This function converges pointwise to zero. Indeed, $(1 + n^2x^2) \sim n^2x^2$ as n gets larger and larger. So,

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{nx}{n^2x^2} = \frac{1}{x} \lim_{n\to\infty} \frac{1}{n} = 0.$$

But for any $\varepsilon < 1/2$, we have

$$\left| f_n\left(\frac{1}{n}\right) - f\left(\frac{1}{n}\right) \right| = \frac{1}{2} - 0 > \varepsilon.$$

Hence $\{f_n\}$ is not uniformly convergent.

Theorem. *Let D be a subset of $\mathbb{R}$ and let $\{f_n\}$ be a sequence of* **continuous** *functions on D which converges uniformly to f on D. Then f is continuous on D*

Homework

Problem 1.
Let $\{f_n\}$ be the sequence of functions on $[0,\ 1]$ defined by $f_n(x) = nx(1-x^4)^n$. Show that $\{f_n\}$ converges pointwise. Find its pointwise limit.

Problem 2.
Is the sequence of functions on $[0,\ 1)$ defined by $f_n(x) = (1-x)^{\frac{1}{n}}$ pointwise convergent? Justify your answer.

Problem 3.
Consider the sequence $\{f_n\}$ of functions defined by

$$f_n(x) = \frac{n + \cos(nx)}{2n+1} \qquad \text{for all } x \text{ in } \mathbb{R}.$$

Show that $\{f_n\}$ is pointwise convergent. Find its pointwise limit.

Problem 4.
Consider the sequence $\{f_n\}$ of functions defined on $[0,\ \pi]$ by $f_n(x) = \sin^n(x)$. Show that $\{f_n\}$ converges pointwise. Find its pointwise limit. Using the above theorem, show that $\{f_n\}$ is not uniformly convergent.