Matrix Methods for Linear Systems of Differential Equations

We now present an application of matrix methods to linear systems of differential equations. We shall follow the development given in Chapter 9 of *Fundamentals of Differential Equations and Boundary Value Problems* by Nagle, Saff, Snider, third edition.

Calculus of Matrices

If we allow the entries $a_{ij}(t)$ in an $n \times n$ matrix $A(t)$ to be functions of the variable t, then $A(t)$ is a *matrix function of* t. Similarly if the entries $x_i(t)$ of a vector $x(t)$ are functions of t, then $x(t)$ is a *vector function of* t. A matrix $A(t)$ is said to be *continuous at* t_0 if each $a_{ij}(t)$ is continuous at t_0. $A(t)$ is *differentiable at* t_0 if each $a_{ij}(t)$ is differentiable at t_0 and we write

$$\frac{dA}{dt}(t_0) = A'(t_0) = \left[a'_{ij}(t_0)\right]_{n\times n}$$

Also

$$\int_a^b A(t)dt = \left[\int_a^b a_{ij}(t)dt\right]_{n\times n}$$

We have the following differentiation formulas for matrices

$$\frac{d}{dt}(CA) = C\frac{dA}{dt}, \quad C \text{ a constant matrix}$$

$$\frac{d}{dt}(A+B) = \frac{dA}{dt} + \frac{dB}{dt}$$

$$\frac{d}{dt}(AB) = A\frac{dB}{dt} + B\frac{dA}{dt}$$

In the last formula the order in which the matrices are written is important, since matrix multiplication need not be commutative.

Linear Systems in Normal Form

A system of n linear differential equations is in *normal form* if it is expressed as

$$x'(t) = A(t)x(t) + f(t) \tag{1}$$

where $x(t)$ and $f(t)$ are $n \times 1$ column vectors and $A(t) = [a_{ij}(t)]_{n\times n}$.

A system is called *homogeneous* if $f(t) = 0$; otherwise it is called *nonhomogeneous*. When the elements of A are constants, the system is said to have *constant coefficients.*

We note that a linear *nth* order differential equation

$$y^{(n)}(t) + p_{n-1}(t)y^{(n-1)} + \cdots + p_0(t)y = g(t) \tag{2}$$

can be rewritten as a first order system in normal form using the substitution

$$x_1(t) = y(t),\ \ x_2(t) = y'(t), \ldots, \ x_n(t) = y^{(n-1)}(t) \tag{2.5}$$

Then

$$\begin{aligned} x_1'(t) &= y'(t) = x_2(t) \\ x_2'(t) &= y''(t) = x_3(t) \\ &\vdots \\ x_{n-1}'(t) &= y^{(n-1)}(t) = x_n(t) \\ x_n'(t) &= y^{(n)}(t) = -p_{n-1}(t)y^{(n-1)} - \cdots - p_0(t)y + g(t) \end{aligned}$$

From (2.5) we can write this last equation as

$$x_n'(t) = -p_0(t)x_1(t) - \cdots - p_{n-1}(t)x_n(t) + g(t)$$

Thus the differential equation (2) can be put in the form (1) with

$$x(t) = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, f(t) = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ g(t) \end{bmatrix}$$

and

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ -p_0(t) & -p_1(t) & -p_2(t) & \cdots & -p_{n-2}(t) & -p_{n-1}(t) \end{bmatrix}$$

The *initial value problem* for the normal system (1) is the problem of finding a differential vector function $x(t)$ that satisfies the system on an interval I and also satisfies the *initial condition* $x(t_0) = x_0$, where t_0 is a given point of I and x_0 is a given constant vector.

Example:
Convert the initial value problem

$$\begin{aligned} y'' + 3y' + 2y &= 0 \\ y(0) &= 1 \\ y'(0) &= 3 \end{aligned}$$

into an initial value problem for a system in normal form.
Solution: $y'' = -3y' - 2y.$ $\quad x_1(t) = y(t) \quad x_2(t) = y'(t)$

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= -3x_2 - 2x_1 \end{aligned}$$

Thus

$$x'(t) = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

We also have the initial condition $x(0) = \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} = x_0.$

Theorem 1 (Existence and Uniqueness)
Suppose $A(t)$ and $f(t)$ are continuous on an open interval I that contains the point t_0. Then, for any choice of the initial vector x_0 there exists a unique solution $x(t)$ on the entire interval I to the initial value problem

$$x'(t) = A(t)x(t) + f(t), \quad x(t_0) = x_0$$

Remark: Just as in Ma 221 we may introduce the *Wronskian* of *n* vectors functions and use it to test for linear independence. We have

Definition: The *Wronskian* of the *n* vector functions

$$x_1(t) = col(x_{11}, x_{21}, \ldots, x_{n1}), \ldots, x_n(t) = col(x_{1n}, x_{2n}, \ldots, x_{nn})$$

is defined to be the real-valued function

$$W[x_1,\ldots,x_n](t) = \begin{vmatrix} x_{11}(t) & x_{12}(t) & \cdots & x_{1n}(t) \\ x_{21}(t) & x_{22}(t) & \cdots & x_{2n}(t) \\ \vdots & \vdots & \vdots & \vdots \\ x_{n1}(t) & x_{n2}(t) & \cdots & x_{nn}(t) \end{vmatrix}$$

One can show that the Wronskian of solutions $x_1,\ldots,x_n$ to $x' = Ax$ is either identically zero or never zero on and interval I. Also, a set of n solutions $x_1,\ldots\ldots,x_n$ to $x' = Ax$ on I is linearly independent if and only if their Wronskian is never zero on I. Thus the Wronskian provides us with an easy test for linear independence for solutions of $x' = Ax$.

Theorem 2 (Representation of Solutions - Homogeneous Case)

Let $x_1, x_2,\ldots, x_n$ be n linearly independent solutions to the homogeneous system

$$x'(t) = A(t)x(t) \tag{3}$$

on the interval I, where $A(t)$ is an $n \times n$ matrix function continuous on I. Then every solution of (3) on I can be expressed in the form

$$x(t) = c_1x_1(t) + \cdots + c_nx_n(t) \tag{4}$$

where $c_1,\ldots, c_n$ are constants.

A set of solutions $\{x_1,\ldots,x_n\}$ that are linearly independent on I is called a *fundamental solution set* for (3). The linear combination (4) is referred to as the general solution of (3).

Exercise:

Verify that $\left\{ \begin{bmatrix} e^{2t} \\ e^{2t} \\ e^{2t} \end{bmatrix}, \begin{bmatrix} -e^{-t} \\ 0 \\ e^{-t} \end{bmatrix}, \begin{bmatrix} -e^{-t} \\ e^{-t} \\ 0 \end{bmatrix} \right\}$ is a fundamental solution set for the system

$$x'(t) = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} x(t) \tag{5}$$

$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$, eigenvectors: $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\} \leftrightarrow -1, \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\} \leftrightarrow 2$

Consider $x_3(t) = \begin{bmatrix} -e^{-t} \\ e^{-t} \\ 0 \end{bmatrix}$. Then

$$Ax_3(t) = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} -e^{-t} \\ e^{-t} \\ 0 \end{bmatrix} = \begin{bmatrix} e^{-t} \\ -e^{-t} \\ 0 \end{bmatrix} = x_3'(t)$$

Remark: The matrix $X(t) = \begin{bmatrix} e^{2t} & -e^{-t} & -e^{-t} \\ e^{2t} & 0 & e^{-t} \\ e^{2t} & e^{-t} & 0 \end{bmatrix}$ is a fundamental matrix for the DE (5). The general solution of (5) can be written as

$$x(t) = X(t)c = c_1\begin{bmatrix} e^{2t} \\ e^{2t} \\ e^{2t} \end{bmatrix} + c_2\begin{bmatrix} -e^{-t} \\ 0 \\ e^{-t} \end{bmatrix} + c_3\begin{bmatrix} -e^{-t} \\ e^{-t} \\ 0 \end{bmatrix}$$

Remark: If we define an operator L by

$$L[x] = x' - Ax$$

then this operator is *linear*. That is, $L[c_1x_1 + c_2x_2] = c_1L[x_1] + c_2L[x_2]$. Thus if x_1 and x_2 are homogeneous solutions of the homogeneous equation

$$x' = Ax$$

the $c_1x_1 + c_2x_2$ is also a solution of this equation. Another consequence of this linearity is the *superposition principle* for linear systems. It states that if x_{p_1} and x_{p_2} are solutions respectively of the *nonhomogeneous* systems $L[x] = g_1$ and $L[x] = g_2$, then $x_{p_1} + x_{p_2}$ is a solution of $L[x] = g_1 + g_2$. This leads to

Theorem 3 (Representation of Solutions - Nonhomogeneous Case)
Let x_p be a particular solution to the nonhomogeneous system

$$x'(t) = A(t)x(t) + f(t) \tag{6}$$

on the interval I, and let $\{x_1, x_2, \ldots, x_n\}$ be a fundamental solution set on I for the corresponding homogeneous system $x'(t) = A(t)x(t)$. Then every solution to (6) on I can be expressed in the form

$$x(t) = x_p(t) + c_1x_1(t) + \cdots + c_nx_n(t) \tag{7}$$

where $c_1, \ldots, c_n$ are constants.

Remark: The linear combination of $x_p, x_1, \ldots., x_n$ written in (7) with arbitrary constants $c_1, \ldots.., c_n$ is called the *general solution* of (6). We may express this solution as $x = x_p + Xc,$ where X is a fundamental matrix for the homogeneous system and c is an arbitrary constant vector.

Solving Normal Systems

1. To determine a general solution to the $n \times n$ homogeneous system $x' = Ax$:

- **a**. Find a fundamental solution set $\{x_1, \ldots.., x_n\}$ that consists of n linearly independent solutions to the homogeneous equation.
- **b**. Form the linear combination

$$x = Xc = c_1 x_1 + \cdots + c_n x_n$$

where $c = col(c_1, \ldots.., c_n)$ is any constant vector and $X = [x_1, \ldots.., x_n]$ is the fundamental matrix, to obtain a general solution.

2. To determine a general solution of to the nonhomogeneous system $x' = Ax + f$:

- **a**. Find a particular solution x_p to the nonhomogeneous system.
- **b**. Form the sum of the particular solution and the general solution $Xc = c_1 x_1 + \cdots + c_n x_n$ to the corresponding homogeneous system in part 1,

$$x = x_p + Xc = x_p + c_1 x_1 + \cdots + c_n x_n$$

to obtain a general solution.

Homogeneous Linear Systems with Constant Coefficients

Consider now the system

$$x'(t) = Ax(t) \tag{8}$$

where A is a (real) *constant* $n \times n$ matrix.

Theorem 4

Suppose the $n \times n$ constant matrix A has n linearly independent eigenvectors $u_1, u_2, \ldots, u_n$. Let r_i be the eigenvalue corresponding to the u_i. Then

$$\{e^{r_1 t} u_1, e^{r_2 t} u_2, \ldots., e^{r_n t} u_n\} \tag{9}$$

is a fundamental solution set on $(-\infty, \infty)$ for the homogeneous system $x' = Ax$. Hence the general solution of $x' = Ax$ is

$$x(t) = c_1 e^{r_1 t} u_1 + \cdots + c_n e^{r_n t} u_n$$

where $c_1, \ldots., c_n$ are arbitrary constants.

Remark: The eigenvalues may be real or complex and need not be distinct.

Proof
Since $Au_i = r_i u_i$ we have

$$\frac{d}{dt}(e^{r_i t}u_i) = r_i e^{r_i t}u_i = e^{r_i t}Au_i = A(e^{r_i t}u_i)$$

so each element of the set (9) is a solution of the system (8). Also the Wronskian of these solutions is

$$W(t) = \det[e^{r_1 t}u_1, \ldots\ldots, e^{r_n t}u_n] = e^{(r_1+r_2+\cdots r_n)t}\det[u_1, \ldots\ldots, u_n] \neq 0$$

since the eigenvectors are linearly independent.

Example
Find a general solution of

$$x' = \begin{bmatrix} 5 & 4 \\ -1 & 0 \end{bmatrix} x$$

$\begin{bmatrix} 5 & 4 \\ -1 & 0 \end{bmatrix}$, eigenvectors: $\left\{\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right\} \leftrightarrow 1, \left\{\begin{bmatrix} -4 \\ 1 \end{bmatrix}\right\} \leftrightarrow 4$

Thus $x(t) = c_1 e^t \begin{bmatrix} -1 \\ 1 \end{bmatrix} + c_2 e^{4t}\begin{bmatrix} -4 \\ 1 \end{bmatrix}$
Thus the solution is

$$\begin{aligned} x_1(t) &= -c_1 e^t - 4c_2 e^{4t} \\ x_2(t) &= c_1 e^t + c_2 e^{4t} \end{aligned} \qquad (*)$$

SNB gives the following strange looking result:

$\begin{bmatrix} x_1' = 5x_1 + 4x_2 \\ x_2' = -x_1 \end{bmatrix}$, Exact solution is: $\begin{bmatrix} x_1(t) = -\frac{1}{3}C_1 e^t + \frac{4}{3}C_1 e^{4t} + \frac{4}{3}C_2 e^{4t} - \frac{4}{3}C_2 e^t \\ x_2(t) = -\frac{1}{3}C_1 e^{4t} + \frac{1}{3}C_1 e^t + \frac{4}{3}C_2 e^t - \frac{1}{3}C_2 e^{4t} \end{bmatrix}$

This is correct and is equivalent to $(*)$, if we let $c_1 = \frac{1}{3}C_1 + \frac{4}{3}C_2$ and $c_2 = -\left(\frac{1}{3}C_1 + \frac{1}{3}C_2\right)$.
However, it is a most cumbersome form of the solution.

Exercise:
Nagle and Saff page 535 #23. Find a fundamental matrix for the system

$$x'(t) = \begin{bmatrix} 2 & 1 & 1 & -1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 7 \end{bmatrix} x(t)$$

Solution:

$$\begin{bmatrix} 2 & 1 & 1 & -1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 7 \end{bmatrix}, \text{ eigenvectors: } \left\{\begin{bmatrix} 1 \\ -3 \\ 0 \\ 0 \end{bmatrix}\right\} \leftrightarrow -1, \left\{\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right\} \leftrightarrow 2, \left\{\begin{bmatrix} -1 \\ 1 \\ 2 \\ 8 \end{bmatrix}\right\} \leftrightarrow 7, \left\{\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}\right\} \leftrightarrow 3$$

Hence the four linearly independent solutions are

$$e^{-t}\begin{bmatrix} 1 \\ -3 \\ 0 \\ 0 \end{bmatrix}, e^{2t}\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, e^{7t}\begin{bmatrix} -1 \\ 1 \\ 2 \\ 8 \end{bmatrix}, e^{3t}\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

Therefore a fundamental matrix is

$$\begin{bmatrix} e^{-t} & e^{2t} & -e^{7t} & e^{3t} \\ -3e^{-t} & 0 & e^{7t} & 0 \\ 0 & 0 & 2e^{7t} & e^{3t} \\ 0 & 0 & 8e^{7t} & 0 \end{bmatrix}$$

We know that if a matrix has n distinct eigenvalues, then the eigenvectors associated with these eigenvalues are linearly independent. Hence

Corollary

If the $n \times n$ constant matrix A has n distinct eigenvalues $r_1, \ldots, r_n$ and u_i is an eigenvector associated with r_i then $\{e^{r_1 t}u_1, \ldots, e^{r_n t}u_n\}$ is a fundamental solution set for the homogeneous system $x' = Ax$.

Example
Solve the initial value problem

$$x'(t) = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 4 & -4 & 5 \end{bmatrix} x(t) \quad x(0) = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$$

Solution: $\begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 4 & -4 & 5 \end{bmatrix}$, eigenvectors:

$\left\{\begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}\right\} \leftrightarrow 1, \left\{\begin{bmatrix} -2 \\ 1 \\ 4 \end{bmatrix}\right\} \leftrightarrow 2, \left\{\begin{bmatrix} -1 \\ 1 \\ 4 \end{bmatrix}\right\} \leftrightarrow 3$

Thus $x(t) = c_1 e^t \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} -2 \\ 1 \\ 4 \end{bmatrix} + c_3 e^{3t} \begin{bmatrix} -1 \\ 1 \\ 4 \end{bmatrix} = \begin{bmatrix} -c_1 e^t - 2c_2 e^{2t} - c_3 e^{3t} \\ c_1 e^t + c_2 e^{2t} + c_3 e^{3t} \\ 2c_1 e^t + 4c_2 e^{2t} + 4c_3 e^{3t} \end{bmatrix}$

We define $x(t)$ via this. $x(t) = \begin{bmatrix} -c_1 e^t - 2c_2 e^{2t} - c_3 e^{3t} \\ c_1 e^t + c_2 e^{2t} + c_3 e^{3t} \\ 2c_1 e^t + 4c_2 e^{2t} + 4c_3 e^{3t} \end{bmatrix}$ so that

$x(0) = \begin{bmatrix} -c_1 - 2c_2 - c_3 \\ c_1 + c_2 + c_3 \\ 2c_1 + 4c_2 + 4c_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$

Thus we form $\begin{bmatrix} -1 & -2 & -1 & -1 \\ 1 & 1 & 1 & 0 \\ 2 & 4 & 4 & 0 \end{bmatrix}$, row echelon form: $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$. Hence $c_1 = 0, c_2 = 1, c_3 = -1.$ Then the solution is

$$x(t) = e^{2t} \begin{bmatrix} -2 \\ 1 \\ 4 \end{bmatrix} - e^{3t} \begin{bmatrix} -1 \\ 1 \\ 4 \end{bmatrix}$$

Complex Eigenvalues

We now discuss how one solves the system

$$x'(t) = Ax(t) \qquad (*)$$

in the case where A is a real matrix and the eigenvalues are complex. We shall show how to obtain two real vector solutions of the system $(*)$. Recall that if $r_1 = \alpha + i\beta$ is a solution of the equation that determines the eigenvalues, namely,

$$p(\lambda) = \det(A - rI) = 0$$

then $r_2 = \alpha - i\beta$ is also a solution of this equation, and hence is an eigenvalue. Recall that r_2 is called the complex conjugate of r_1 and $\bar{r}_1 = r_2$.

Let $\mathbf{z} = \mathbf{a} + i\mathbf{b},$ where $\mathbf{a}$ and $\mathbf{b}$ are real vectors, be an eigenvector corresponding to r_1. Then it is not hard to see that $\bar{\mathbf{z}} = \mathbf{a} - i\mathbf{b}$ is an eigenvector corresponding to r_2. Since

$$(A - r_1 I)\mathbf{z} = 0$$

then taking the conjugate of this equation and noting that since A and I are real matrices then $\bar{A} = A$ and $\bar{I} = I$

$$\overline{(A - r_1 I)\mathbf{z}} = (A - \bar{r}_1 I)\bar{\mathbf{z}} = (A - r_2 I)\bar{\mathbf{z}} = 0$$

so $\bar{\mathbf{z}}$ is an eigenvector corresponding to r_2. Therefore the vectors

$$\mathbf{w}_1(t) = e^{(\alpha + i\beta)t}(\mathbf{a} + i\mathbf{b})$$

and

$$\mathbf{w}_2(t) = e^{(\alpha - i\beta)t}(\mathbf{a} - i\mathbf{b})$$

are two linearly independent vector solutions of $(*)$. However, they are not real. To get real solutions we proceed as follows: Since

$$e^{(\alpha + i\beta)t} = e^{\alpha t}(\cos\beta t + i\sin\beta t)$$

then

$$\mathbf{w}_1(t) = e^{(\alpha + i\beta)t}(\mathbf{a} + i\mathbf{b}) = e^{\alpha t}\{(\cos\beta t\mathbf{a} - \sin\beta t\mathbf{b}) + i(\sin\beta t\mathbf{a} + \cos\beta t\mathbf{b})\}$$

Therefore

$$\mathbf{w}_1(t) = \mathbf{x}_1(t) + i\mathbf{x}_2(t)$$

where

$$\mathbf{x}_1(t) = e^{\alpha t}(\cos\beta t\mathbf{a} - \sin\beta t\mathbf{b})$$
$$\mathbf{x}_2(t) = e^{\alpha t}(\sin\beta t\mathbf{a} + \cos\beta t\mathbf{b})$$

Since $\mathbf{w}_1(t)$ is a solution of $(*)$, then

$$\mathbf{w}_1'(t) = A\mathbf{w}_1(t)$$

so

$$\mathbf{x}_1'(t) + i\mathbf{x}_2'(t) = A\mathbf{x}_1(t) + iA\mathbf{x}_2(t)$$

Equating real and imaginary parts of this last equation leads to the real equations

$$\mathbf{x}_1'(t) = A\mathbf{x}_1(t) \qquad \mathbf{x}_2'(t) = A\mathbf{x}_2(t)$$

so that $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ are real vector solutions of $(*)$ corresponding to the eigenvalues $\alpha \pm i\beta$. Note that we can get the two expressions above for $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ by taking the real and imaginary parts of $\mathbf{w}_1(t)$

Example

Find the general solution of the initial value problem

$$x'(t) = \begin{bmatrix} -3 & -1 \\ 2 & -1 \end{bmatrix} x(t) \qquad x\left(\frac{\pi}{2}\right) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Solution:

We first find the eigenvalues and eigenvectors of the matrix. We want the roots of

$$\begin{vmatrix} -3-r & -1 \\ 2 & -1-r \end{vmatrix} = (3+r)(1+r)+2 = r^2+4r+5 = 0$$

Thus

$$r = \frac{-4 \pm \sqrt{16-4(1)(5)}}{2} = -2 \pm i$$

The system of equations to determine the eigenvectors is

$$(-3-r)x_1 - x_2 = 0$$
$$2x_1 + (-1-r)x_2 = 0$$

or

$$(3+r)x_1 + x_2 = 0$$
$$2x_1 + (-1-r)x_2 = 0$$

For $r = -2+i$ we have

$$(1+i)x_1 + x_2 = 0$$
$$2x_1 + (1-i)x_2 = 0$$

Multiplication of the first equation by $i - i$ yields the second equation, since $(1 = i)(1-i) = 2$. Thus

$$x_2 = -(1+i)x_1$$

Letting $x_1 = 1$ gives the eigenvector $\begin{bmatrix} 1 \\ -1-i \end{bmatrix}$. Since the second eigenvector is the complex conjugate of the first we have

$\begin{bmatrix} -3 & -1 \\ 2 & -1 \end{bmatrix}$, eigenvectors: $\left\{\begin{bmatrix} 1 \\ -1-i \end{bmatrix}\right\} \leftrightarrow -2+i, \left\{\begin{bmatrix} 1 \\ -1+i \end{bmatrix}\right\} \leftrightarrow -2-i$

$$\begin{bmatrix} 1 \\ -1-i \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} + i\begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

Thus $\alpha = -2, \beta = 1, \mathbf{a} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$. The two linearly independent solutions are

$$w(t) = e^{(-2+i)t}\begin{bmatrix} 1 \\ -1-i \end{bmatrix} = \begin{bmatrix} 1e^{-2t}e^{it} \\ (-1-i)e^{-2t}e^{it} \end{bmatrix}$$

$$= \begin{bmatrix} e^{-2t}(\cos t + i\sin t) \\ (-1-i)e^{-2t}(\cos t + i\sin t) \end{bmatrix} = \begin{bmatrix} e^{-2t}\cos t + ie^{-t}\sin t \\ e^{-2t}(-\cos t + \sin t) + ie^{-2t}(-\cos t - \sin t) \end{bmatrix}$$

$$= e^{-2t}\left(\cos t\begin{bmatrix} 1 \\ -1 \end{bmatrix} - \sin t\begin{bmatrix} 0 \\ -1 \end{bmatrix}\right) + ie^{-2t}\left(\sin t\begin{bmatrix} 1 \\ -1 \end{bmatrix} + \cos t\begin{bmatrix} 0 \\ -1 \end{bmatrix}\right)$$

Therefore

$$x_1(t) = e^{\alpha t}(\cos\beta t\mathbf{a} - \sin\beta t\mathbf{b}) = e^{-2t}\left(\cos t\begin{bmatrix} 1 \\ -1 \end{bmatrix} - \sin t\begin{bmatrix} 0 \\ -1 \end{bmatrix}\right)$$

$$x_2(t) = e^{\alpha t}(\sin\beta t\mathbf{a} + \cos\beta t\mathbf{b}) = e^{-2t}\left(\sin t\begin{bmatrix} 1 \\ -1 \end{bmatrix} + \cos t\begin{bmatrix} 0 \\ -1 \end{bmatrix}\right)$$

Thus

$$x(t) = c_1e^{-2t}\left(\cos t\begin{bmatrix} 1 \\ -1 \end{bmatrix} - \sin t\begin{bmatrix} 0 \\ -1 \end{bmatrix}\right) + c_2e^{-2t}\left(\sin t\begin{bmatrix} 1 \\ -1 \end{bmatrix} + \cos t\begin{bmatrix} 0 \\ -1 \end{bmatrix}\right)$$

$x\left(\frac{\pi}{2}\right) = -c_1e^{-\pi}\begin{bmatrix} 0 \\ -1 \end{bmatrix} + c_2e^{-\pi}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Therefore

$$c_1 = e^{\pi} \text{ and } c_2 = 0$$

so

$$x(t) = e^{\pi-2t}\left(\cos t\begin{bmatrix} 1 \\ -1 \end{bmatrix} - \sin t\begin{bmatrix} 0 \\ -1 \end{bmatrix}\right)$$

Example Find a [real] general solution to

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Solution: We first find the eigenvalues and eigenvectors of the matrix. We want the solutions to

$$\det\begin{bmatrix} 3-r & 1 \\ -2 & 1-r \end{bmatrix} = (3-r)(1-r) + 2 = r^2 - 4r + 5 = 0$$

Thus

$$r = \frac{4 \pm \sqrt{16 - 4(1)(5)}}{2} = 2 \pm i$$

The system of equations to determine the eigenvectors is

$$(3-r)x_1 + x_2 = 0$$
$$-2x_1 + (1-r)x_2 = 0$$

For $r = 2+i$ we have

$$(1-i)x_1 + x_2 = 0$$
$$-2x_1 + (-1-i)x_2 = 0$$

or

$$(1-i)x_1 + x_2 = 0$$
$$2x_1 + (1+i)x_2 = 0$$

One can see that the first and second equations are the same by multiplying the first equation by $1+i$ and recalling that $(1+i)(1-i) = 2$. Thus

$$x_2 = -\frac{1}{1-i}x_1$$

We have

$$x_1 = -\frac{1}{1-i}x_2 = -\left(\frac{1}{1-i}\right)\left(\frac{1+i}{1+i}\right)x_2 = -\left(\frac{1+i}{2}\right)x_2$$

Letting $x_2 = 2$ we have the eigenvector

$$\begin{bmatrix} -1-i \\ 2 \end{bmatrix}$$

Since the eigenvectors are complex conjugates we have that the eigenvalues of the matrix $\begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix}$ are $2+i$ and $2-i$ and the corresponding eigenvectors are $\begin{bmatrix} -1-i \\ 2 \end{bmatrix}$ and $\begin{bmatrix} -1+i \\ 2 \end{bmatrix}$.

The eigenvalues of the matrix $\begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix}$ are $2+i$ and $2-i$ and the corresponding eigenvectors are $\begin{bmatrix} -1-i \\ 2 \end{bmatrix}$ and $\begin{bmatrix} -1+i \\ 2 \end{bmatrix}$.

Let $r_1 = 2+i$, so $\alpha = 2, \beta = 1$. Also

$$\begin{bmatrix} -1-i \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix} + i\begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

so $\mathbf{a} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$.

$$x_1(t) = e^{\alpha t}(\cos\beta t\mathbf{a} - \sin\beta t\mathbf{b}) = e^{2t}\left(\cos t\begin{bmatrix} -1 \\ 2 \end{bmatrix} - \sin t\begin{bmatrix} -1 \\ 0 \end{bmatrix}\right)$$

$$x_2(t) = e^{\alpha t}(\sin\beta t\mathbf{a} + \cos\beta t\mathbf{b}) = e^{2t}\left(\sin t\begin{bmatrix} -1 \\ 2 \end{bmatrix} + \cos t\begin{bmatrix} -1 \\ 0 \end{bmatrix}\right)$$

$$x(t) = c_1 e^{2t}\left(\cos t\begin{bmatrix} -1 \\ 2 \end{bmatrix} - \sin t\begin{bmatrix} -1 \\ 0 \end{bmatrix}\right) + c_2 e^{2t}\left(\sin t\begin{bmatrix} -1 \\ 2 \end{bmatrix} + \cos t\begin{bmatrix} -1 \\ 0 \end{bmatrix}\right)$$

$$= c_1\begin{bmatrix} e^{2t}(-\cos t + \sin t) \\ 2e^{2t}\cos t \end{bmatrix} + c_2\begin{bmatrix} e^{2t}(-\sin t - \cos t) \\ 2e^{2t}\sin t \end{bmatrix}$$

Nonhomogeneous Linear Systems

The techniques of Undetermined Coefficients and Variation of Parameters that are used to find particular solutions to the nonhomogeneous equation

$$y'' + p(x)y' + q(x)y = g(x)$$

have analogies to nonhomogeneous systems. Thus we now discuss how one solves the nonhomogeneous system

$$x'(t) = A(t)x(t) + f(t)$$

Undetermined Coefficients

Consider the nonhomogeneous *constant* coefficient system

$$x'(t) = Ax(t) + f(t)$$

Before presenting the method for systems we recall the following result for second nonhomogeneous order differential equations. For more on this see Linear Second Order DEs (Hold down the Shift key and click.)

A particular solution of

$$ay'' + by' + cy = Ke^{\alpha x}$$

where a, b, c are constants is

$$y_p = \frac{Ke^{\alpha x}}{p(\alpha)} \quad \text{if } p(\alpha) \neq 0$$

$$y_p = \frac{Kxe^{\alpha x}}{p'(\alpha)} \quad \text{if } p(\alpha) = 0, \quad p'(\alpha) \neq 0$$

$$y_p = \frac{K}{p''(\alpha)}x^2 e^{\alpha x} \quad \text{if } p(\alpha) = p'(\alpha) = 0$$

where

$$p(r) = ar^2 + br + c$$

Example

$$y'' - 5y' + 4y = 2e^x$$

Homogeneous solution: $p(\lambda) = \lambda^2 - 5\lambda + 4 = (\lambda - 4)(\lambda - 1) \quad \Rightarrow \lambda = 4,\ 1 \Rightarrow y_h = c_1 e^x + c_2 e^{4x}$
Now to find a particular solution for $2e^x$. $\Rightarrow \alpha = 1 \quad p(1) = 0 \quad$ Since $p'(\lambda) = 2\lambda - 5$
$p'(1) = 2 - 5 = -3 \neq 0$
$\Rightarrow$

$$y_p = \frac{kxe^{\alpha x}}{p'(\alpha)} = \frac{2xe^x}{-3}$$

$\Rightarrow$

$$y = y_h + y_p = c_1 e^x + c_2 e^{4x} - \frac{2}{3} xe^x$$

Example Problem 3, page 547 of text.
Find the general solution of

$$x'(t) = \begin{bmatrix} 1 & -2 & 2 \\ -2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} x(t) + \begin{bmatrix} 2e^t \\ 4e^t \\ -2e^t \end{bmatrix}$$

Solution:
We first find the homogeneous solution.

$$\begin{bmatrix} 1 & -2 & 2 \\ -2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}, \text{ eigenvectors: } \left\{ \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \right\} \leftrightarrow -3, \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\} \leftrightarrow 3$$

Since these eigenvectors are linearly independent, then

$$x_h(t) = c_1 e^{-3t} \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} + c_2 e^{3t} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + c_3 e^{3t} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

We seek a particular solution of the form

$$x_p(t) = e^t \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

Then

$$x_p'(t) = e^t \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = Ax_p(t) + \begin{bmatrix} 2e^t \\ 4e^t \\ -2e^t \end{bmatrix} = e^t \begin{bmatrix} 1 & -2 & 2 \\ -2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} + \begin{bmatrix} 2e^t \\ 4e^t \\ -2e^t \end{bmatrix}$$

$$= e^t \left(\begin{bmatrix} a_1 - 2a_2 + 2a_3 \\ -2a_1 + a_2 + 2a_3 \\ 2a_1 + 2a_2 + a_3 \end{bmatrix} + \begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix} \right)$$

Thus

$$
\begin{aligned}
a_1 &= a_1 - 2a_2 + 2a_3 + 2 \\
a_2 &= -2a_1 + a_2 + 2a_3 + 4 \\
a_3 &= 2a_1 + 2a_2 + a_3 - 2
\end{aligned}
$$

, Solution is: $\{a_2 = 0, a_1 = 1, a_3 = -1\}$

Therefore

$$x_p(t) = e^t \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

and

$$x(t) = x_h(t) + x_p(t) = c_1 e^{-3t} \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} + c_2 e^{3t} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + c_3 e^{3t} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + e^t \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

Note: the Method of Undetermined Coefficients works only for constant coefficient systems.

Example Problem 2, Page 547 of text. Find a general solution to

$$x'(t) = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} x(t) + \begin{bmatrix} -t-1 \\ -4t-2 \end{bmatrix}$$

Solution: We first find a general homogeneous solution.

$$\begin{vmatrix} 1-r & 1 \\ 4 & 1-r \end{vmatrix} = (1-r)^2 - 4 = r^2 - 2r - 3 = (r-3)(r+1)$$

Thus the eigenvalues are $r = -1, 3$. The equations that determine the eigenvectors are

$$
\begin{aligned}
(1-r)x_1 + x_2 &= 0 \\
4x_1 + (1-r)x_2 &= 0
\end{aligned}
$$

For $r = 3$ we have

$$
\begin{aligned}
-2x_1 + x_2 &= 0 \\
4x_1 - 3x_2 &= 0
\end{aligned}
$$

Thus $x_2 = 2x_1$ and an eigenvector is $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

For $r = -1$ we have

$$
\begin{aligned}
2x_1 + x_2 &= 0 \\
4x_1 + 2x_2 &= 0
\end{aligned}
$$

Thus $x_1 = -\frac{1}{2}x_2$ and we have the eigenvector $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$. Hence

$$x_h(t) = c_1 e^{3t}\begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 e^{-t}\begin{bmatrix} -1 \\ 2 \end{bmatrix}.$$

To find x_p we let

$$x_p(t) = \begin{bmatrix} a_1 t + b_1 \\ a_2 t + b_2 \end{bmatrix}$$

since $f(t)$ is a polynomial.

Plugging into the DE we have

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} a_1 t + b_1 \\ a_2 t + b_2 \end{bmatrix} + \begin{bmatrix} -t-1 \\ -4t-2 \end{bmatrix}$$

$$= \begin{bmatrix} b_1 - t + b_2 + ta_1 + ta_2 - 1 \\ 4b_1 - 4t + b_2 + 4ta_1 + ta_2 - 2 \end{bmatrix}$$

Equating the coefficients of t on both sides we have

$$0 = -1 + a_1 + a_2$$
$$0 = -4 + 4a_1 + a_2$$

or

$$a_1 + a_2 = 1$$
$$4a_1 + a_2 = 4$$

Therefore $a_1 = 1, a_2 = 0$.
Equating the constant terms on both sides we have

$$a_1 = b_1 + b_2 - 1$$
$$a_2 = 4b_1 + b_2 - 2$$

Using the values for a_1, a_2 we have

$$b_1 + b_2 = 2$$
$$4b_1 + b_2 = 2$$

Thus $b_1 = 0, b_2 = 2.$ With these values for the constants we have that

$$x_p(t) = \begin{bmatrix} t \\ 2 \end{bmatrix}$$

Finally

$$x(t) = x_h(t) + x_p(t) = c_1 e^{3t}\begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 e^{-t}\begin{bmatrix} -1 \\ 2 \end{bmatrix} + \begin{bmatrix} t \\ 2 \end{bmatrix}$$

Example Problem 4, page 547 of text. Find a general solution to

$$x'(t) = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} x(t) + \begin{bmatrix} -4\cos t \\ -\sin t \end{bmatrix}$$

Solution: We first find a homogeneous solution.

$$\begin{vmatrix} 2-r & 2 \\ 2 & 2-r \end{vmatrix} = (2-r)^2 - 4 = 4 - 4r + r^2 - r = r^2 - 4r = r(r-4)$$

Thus the eigenvalues are $r = 0, 4$.

The equations for the eigenvectors are

$$(2-r)x_1 + 2x_2 = 0$$
$$2x_1 + (2-r)x_2 = 0$$

For $r = 0$ we have

$$2x_1 + 2x_2 = 0$$

or $x_1 = -x_2$. Thus we have the eigenvector $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$. For $r = 4$ we have

$$-2x_1 + 2x_2 = 0$$
$$2x_1 - 2x_2 = 0$$

or $x_1 = x_2$ Thus we have the eigenvector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Hence

$$x_h(t) = c_1 e^{0t} \begin{bmatrix} -1 \\ 1 \end{bmatrix} + c_2 e^{4t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

We assume

$$x_p(t) = \begin{bmatrix} a_1 \cos t + b_1 \sin t \\ a_2 \cos t + b_2 \sin t \end{bmatrix}$$

Hence

$$x_p' = \begin{bmatrix} -a_1 \sin t + b_1 \cos t \\ -a_2 \sin t + b_2 \cos t \end{bmatrix}$$

Plugging into the DE yields

$$\begin{bmatrix} -a_1 \sin t + b_1 \cos t \\ -a_2 \sin t + b_2 \cos t \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} a_1 \cos t + b_1 \sin t \\ a_2 \cos t + b_2 \sin t \end{bmatrix} + \begin{bmatrix} -4\cos t \\ -\sin t \end{bmatrix}$$

or

$$\begin{bmatrix} -a_1 \sin t + b_1 \cos t \\ -a_2 \sin t + b_2 \cos t \end{bmatrix} = \left[\begin{matrix} 2a_1 \cos t + 2a_2 \cos t + 2b_1 \sin t + 2b_2 \sin t \\ 2a_1 \cos t + 2a_2 \cos t + 2b_1 \sin t + 2b_2 \sin t \end{matrix} + \begin{bmatrix} -4\cos t \\ -\sin t \end{bmatrix} \right]$$

We equate the coefficients of the $\sin t$ and $\cos t$ terms. Thus

$$\begin{aligned} b_1 &= 2a_1 + 2a_2 - 4 \\ b_2 &= 2a_1 + 2a_2 \\ -a_1 &= 2b_1 + 2b_2 \\ -a_2 &= 2b_1 + 2b_2 - 1 \end{aligned}$$

or

$$\begin{aligned} 2a_1 + 2a_2 - b_1 &= 4 \\ 2a_1 + 2a_2 - b_2 &= 0 \\ a_1 + 2b_1 + 2b_2 &= 0 \\ a_2 + 2b_1 + 2b_2 &= 1 \end{aligned}$$

To solve this system we form

$$\begin{bmatrix} 1 & 0 & 2 & 2 & 0 \\ 0 & 1 & 2 & 2 & 1 \\ 2 & 2 & -1 & 0 & 4 \\ 2 & 2 & 0 & -1 & 0 \end{bmatrix}$$

and row reduce.

$$\begin{bmatrix} 1 & 0 & 2 & 2 & 0 \\ 0 & 1 & 2 & 2 & 1 \\ 2 & 2 & -1 & 0 & 4 \\ 2 & 2 & 0 & -1 & 0 \end{bmatrix} \xrightarrow{\substack{-2R_1+R_3 \\ -2R_1+R_4}} \begin{bmatrix} 1 & 0 & 2 & 2 & 0 \\ 0 & 1 & 2 & 2 & 1 \\ 0 & 2 & -5 & -4 & 4 \\ 0 & 2 & -4 & -5 & 0 \end{bmatrix} \xrightarrow{\substack{-2R_2+R_3 \\ -2R_2+R_4}} \begin{bmatrix} 1 & 0 & 2 & 2 & 0 \\ 0 & 1 & 2 & 2 & 1 \\ 0 & 0 & -9 & -8 & 2 \\ 0 & 0 & -8 & -9 & -2 \end{bmatrix}$$

$$\xrightarrow{-R_3} \begin{bmatrix} 1 & 0 & 2 & 2 & 0 \\ 0 & 1 & 2 & 2 & 1 \\ 0 & 0 & 9 & 8 & -2 \\ 0 & 0 & -8 & -9 & -2 \end{bmatrix} \xrightarrow{R_3+R_4} \begin{bmatrix} 1 & 0 & 2 & 2 & 0 \\ 0 & 1 & 2 & 2 & 1 \\ 0 & 0 & 9 & 8 & -2 \\ 0 & 0 & 1 & -1 & -4 \end{bmatrix} \xrightarrow{R_4 \leftrightarrow R_3} \begin{bmatrix} 1 & 0 & 2 & 2 & 0 \\ 0 & 1 & 2 & 2 & 1 \\ 0 & 0 & 1 & -1 & -4 \\ 0 & 0 & 9 & 8 & -2 \end{bmatrix}$$

$$\xrightarrow{-9R_3+R_4} \begin{bmatrix} 1 & 0 & 2 & 2 & 0 \\ 0 & 1 & 2 & 2 & 1 \\ 0 & 0 & 1 & -1 & -4 \\ 0 & 0 & 0 & 17 & 34 \end{bmatrix} \xrightarrow{\frac{1}{17}R_4} \begin{bmatrix} 1 & 0 & 2 & 2 & 0 \\ 0 & 1 & 2 & 2 & 1 \\ 0 & 0 & 1 & -1 & -4 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} \xrightarrow{R_4+R_3} \begin{bmatrix} 1 & 0 & 2 & 2 & 0 \\ 0 & 1 & 2 & 2 & 1 \\ 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

$$\xrightarrow{\substack{-2R_3+R_2 \\ -2R_3+R_1}} \begin{bmatrix} 1 & 0 & 0 & 2 & 4 \\ 0 & 1 & 0 & 2 & 5 \\ 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} \xrightarrow{\substack{-2R_4+R_2 \\ -2R_4+R_1}} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

Thus $a_1 = 0, a_2 = 1, b_1 = -2, b_2 = 2$ and

$$x_p(t) = \begin{bmatrix} -2\sin t \\ \cos t + 2\sin t \end{bmatrix}$$

Finally a general solution is

$$x(t) = x_h(t) + x_p(t) = c_1\begin{bmatrix} -1 \\ 1 \end{bmatrix} + c_2 e^{4t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -2\sin t \\ \cos t + 2\sin t \end{bmatrix}$$

Example The eigenvalues of the matrix $\begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix}$ are $2+i$ and $2-i$ and the corresponding eigenvectors are $\begin{bmatrix} -1-i \\ 2 \end{bmatrix}$ and $\begin{bmatrix} -1+i \\ 2 \end{bmatrix}$.

Find a [real] general solution to

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 25t \\ 0 \end{bmatrix}.$$

Solution: First we find a real general solution to $\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

Here $\alpha = 2, \beta = 1$ and $\begin{bmatrix} -1-i \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix} + i\begin{bmatrix} -1 \\ 0 \end{bmatrix} = \mathbf{a} + i\mathbf{b}$.

Since

$$\mathbf{x}_1(t) = e^{\alpha t}(\cos\beta t\mathbf{a} - \sin\beta t\mathbf{b})$$
$$\mathbf{x}_2(t) = e^{\alpha t}(\sin\beta t\mathbf{a} + \cos\beta t\mathbf{b})$$

then

$$\mathbf{x}_1(t) = e^{2t}\left(\cos t\begin{bmatrix} -1 \\ 2 \end{bmatrix} - \sin t\begin{bmatrix} -1 \\ 0 \end{bmatrix}\right)$$
$$\mathbf{x}_2(t) = e^{2t}\left(\sin t\begin{bmatrix} -1 \\ 2 \end{bmatrix} + \cos t\begin{bmatrix} -1 \\ 0 \end{bmatrix}\right)$$

Hence

$$\begin{aligned}\mathbf{x}_h(t) &= c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) \\ &= c_1\begin{bmatrix} e^{2t}(-\cos t + \sin t) \\ 2e^{2t}\cos t \end{bmatrix} + c_2\begin{bmatrix} e^{2t}(-\sin t - \cos t) \\ 2e^{2t}\sin t \end{bmatrix}\end{aligned}$$

Or we may expand one of the complex solutions and take the real and imaginary parts.

$$e^{(2+i)t}\begin{bmatrix} -1-i \\ 2 \end{bmatrix} = e^{2t}(\cos t + i\sin t)\begin{bmatrix} -1-i \\ 2 \end{bmatrix}$$
$$= e^{2t}\begin{bmatrix} (-\cos t + \sin t) + i(-\sin t - \cos t) \\ (2\cos t) + i(2\sin t)) \end{bmatrix}$$
$$= \begin{bmatrix} e^{2t}(-\cos t + \sin t) \\ 2e^{2t}\cos t \end{bmatrix} + i\begin{bmatrix} e^{2t}(-\sin t - \cos t) \\ 2e^{2t}\sin t \end{bmatrix}$$

$$\mathbf{x}_h = c_1\begin{bmatrix} e^{2t}(-\cos t + \sin t) \\ 2e^{2t}\cos t \end{bmatrix} + c_2\begin{bmatrix} e^{2t}(-\sin t - \cos t) \\ 2e^{2t}\sin t \end{bmatrix}$$
$$= \begin{bmatrix} e^{2t}(-\cos t + \sin t) & e^{2t}(-\sin t - \cos t) \\ 2e^{2t}\cos t & 2e^{2t}\sin t \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

Next, we find a particular solution to the given non-homogeneous equation. Since the non-homogeneous term is a polynomial of degree one, the solution must be the same. Thus let

$$x_p(t) = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} at+b \\ ct+d \end{bmatrix}$$

We substitute into the system of DEs and find the coefficients.

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 25t \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} a \\ c \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} at+b \\ ct+d \end{bmatrix} + \begin{bmatrix} 25t \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} a \\ c \end{bmatrix} = \begin{bmatrix} (3a+c)t + (3b+d) \\ (-2a+c)t + (-2b+d) \end{bmatrix} + \begin{bmatrix} 25t \\ 0 \end{bmatrix}$$

We equate like terms.

$$\begin{aligned} 0 &= 3a + c + 25 \\ 0 &= -2a + c \\ a &= 3b + d \\ c &= -2b + d \end{aligned}$$

Thus (from the first pair of equations) $a = -5$, $c = -10$ and then $b = 1$ and $d = -8$. Combining homogeneous and particular solutions, we have a general solution.

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = c_1 \begin{bmatrix} e^{2t}(-\cos t + \sin t) \\ 2e^{2t}\cos t \end{bmatrix} + c_2 \begin{bmatrix} e^{2t}(-\sin t - \cos t) \\ 2e^{2t}\sin t \end{bmatrix} + \begin{bmatrix} -5t+1 \\ -10t-8 \end{bmatrix}$$

$$= \begin{bmatrix} e^{2t}(-\cos t + \sin t) & e^{2t}(-\sin t - \cos t) \\ 2e^{2t}\cos t & 2e^{2t}\sin t \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} + \begin{bmatrix} -5t+1 \\ -10t-8 \end{bmatrix}$$

Example a) Find the eigenvalues and eigenvectors of

$$A = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}.$$

Solution: We solve $\det(A - rI) = 0$.

$$\det(A - rI) = \begin{vmatrix} 2-r & -1 \\ 1 & 2-r \end{vmatrix}$$

$$= (2-r)^2 + 1$$

$$(2-r)^2 = -1$$

$$2 - r = \pm i$$

$$r = 2 \pm i$$

So, the eigenvalues are a complex conjugate pair. We find the eigenvector for one and take the complex conjugate to get the other. For $r = 2 + i$, we solve

$$(A - rI)u = 0$$

$$\begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Thus we have the equations

$$-iu_1 - u_2 = 0$$

$$u_1 - iu_2 = 0$$

The second row is redundant, so $-iu_1 - u_2 = 0$ or $u_2 = -i \cdot u_1$. Hence any multiple of $\begin{bmatrix} 1 \\ -i \end{bmatrix}$ is an eigenvector for $r = 2 + i$. Then an eigenvector corresponding to $r = 2 - i$ is $\begin{bmatrix} 1 \\ i \end{bmatrix}$.

b) Find the [real] general solution to

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 12e^{2t} \end{bmatrix}.$$

Solution: The solution is the general solution (x_h) to the homogeneous equation plus one [particular] solution (x_p) to the full non-homogeneous equation. First we'll find x_p. It is in the form

$$x_p = \begin{bmatrix} c_1 e^{2t} \\ c_2 e^{2t} \end{bmatrix}$$

Substituting into the D.E., we obtain

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 2c_1e^{2t} \\ 2c_2e^{2t} \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} c_1e^{2t} \\ c_2e^{2t} \end{bmatrix} + \begin{bmatrix} 0 \\ 12e^{2t} \end{bmatrix}$$

Hence

$$\begin{bmatrix} 2c_1e^{2t} \\ 2c_2e^{2t} \end{bmatrix} = \begin{bmatrix} 2c_1e^{2t} - c_2e^{2t} \\ c_1e^{2t} + 2c_2e^{2t} \end{bmatrix} + \begin{bmatrix} 0 \\ 12e^{2t} \end{bmatrix} = \begin{bmatrix} 2c_1e^{2t} - c_2e^{2t} \\ c_1e^{2t} + 2c_2e^{2t} + 12e^{2t} \end{bmatrix}$$

We can divide by e^{2t} (which is never zero) and move the unknowns to the left side to obtain

$$\begin{bmatrix} -c_2 \\ -c_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 12 \end{bmatrix}$$

$$x_p = \begin{bmatrix} -12e^{2t} \\ 0 \end{bmatrix}$$

To find a solution to the homogeneous solution we use the eigenvalue $2 + i = \alpha + i\beta$ and the corresponding eigenvector $\begin{bmatrix} 1 \\ -i \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + i\begin{bmatrix} 0 \\ -1 \end{bmatrix} = \mathbf{a} + i\mathbf{b}$.

Since

$$\mathbf{x}_1(t) = e^{\alpha t}(\cos\beta t\mathbf{a} - \sin\beta t\mathbf{b})$$
$$\mathbf{x}_2(t) = e^{\alpha t}(\sin\beta t\mathbf{a} + \cos\beta t\mathbf{b})$$

then

$$\mathbf{x}_1(t) = e^{2t}\left(\cos t\begin{bmatrix} 1 \\ 0 \end{bmatrix} - \sin t\begin{bmatrix} 0 \\ -1 \end{bmatrix}\right)$$
$$\mathbf{x}_2(t) = e^{2t}\left(\sin t\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \cos t\begin{bmatrix} 0 \\ -1 \end{bmatrix}\right)$$

Hence

$$\begin{aligned}\mathbf{x}_h(t) &= c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) \\ &= c_1\begin{bmatrix} e^{2t}\cos t \\ e^{2t}\sin t \end{bmatrix} + c_2\begin{bmatrix} e^{2t}\sin t \\ -e^{2t}\cos t \end{bmatrix}\end{aligned}$$

Or, for the solution to the homogeneous equation, we may use one of the eigenvalues and eigenvectors found in 2a to write a complex solution and break it into real and imaginary parts. we'll use $2 + i$.

$$x = e^{(2+i)t}\begin{bmatrix} 1 \\ -i \end{bmatrix} = e^{2t}(\cos t + i\sin t)\begin{bmatrix} 1 \\ -i \end{bmatrix}$$

$$= \begin{bmatrix} e^{2t}\cos t + ie^{2t}\sin t \\ e^{2t}\sin t - ie^{2t}\cos t \end{bmatrix}$$

$$x_h = c_1\begin{bmatrix} e^{2t}\cos t \\ e^{2t}\sin t \end{bmatrix} + c_2\begin{bmatrix} e^{2t}\sin t \\ -e^{2t}\cos t \end{bmatrix}$$

$$= \begin{bmatrix} e^{2t}\cos t & e^{2t}\sin t \\ e^{2t}\sin t & -e^{2t}\cos t \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

Finally, we add to obtain the desired solution.

$$x = \begin{bmatrix} e^{2t}\cos t & e^{2t}\sin t \\ e^{2t}\sin t & -e^{2t}\cos t \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} + \begin{bmatrix} -12e^{2t} \\ 0 \end{bmatrix}$$

Variation of Parameters (This material is not covered in Ma 227.)

Consider now the nonhomogeneous system

$$x'(t) = A(t)x(t) + f(t) \qquad (1)$$

where the entries in $A(t)$ may be any continuous functions of t. Let $X(t)$ be a fundamental matrix for the homogeneous system

$$x'(t) = A(t)x(t) \qquad (2)$$

The general solution of (2) is

$$X(t)c$$

where c is an $n \times 1$ constant vector. To find a particular solution of (1), consider

$$x_p(t) = X(t)v(t)$$

where $v(t)$ is an $n \times 1$ vector function of t that we wish to determine. Then

$$x_p'(t) = X(t)v'(t) + X'(t)v(t)$$

so that (1) yields

$$X(t)v'(t) + X'(t)v(t) = A(t)x_p(t) + f(t) = A(t)X(t)v(t) + f(t)$$

Since $X(t)$ is a fundamental matrix for (2), then $X'(t) = AX(t)$, so the last equation becomes

$$X(t)v'(t) + A(t)X(t)v(t) = A(t)X(t)v(t) + f(t)$$

or

$$X(t)v'(t) = f(t)$$

Since the columns of $X(t)$ are linearly independent, $X^{-1}(t)$ exists and

$$v'(t) = X^{-1}(t)f(t)$$

Hence

$$v(t) = \int X^{-1}(t) f(t) dt$$

and

$$x_p(t) = X(t)v(t) = X(t)\int X^{-1}(t) f(t) dt$$

Finally the general solution is given by

$$x(t) = x_h(t) + x_p(t) = X(t)c + X(t)\int X^{-1}(t) f(t) dt \qquad (3)$$

Example 2 Page 578

Solve the initial value problem

$$x'(t) = \begin{bmatrix} 2 & -3 \\ 1 & -2 \end{bmatrix} x(t) + \begin{bmatrix} e^{2t} \\ 1 \end{bmatrix}, \quad x(0) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

Solution:

We first find a fundamental matrix for the homogeneous solution.

$$\begin{bmatrix} 2 & -3 \\ 1 & -2 \end{bmatrix}, \text{eigenvectors: } \left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\} \leftrightarrow -1, \left\{\begin{bmatrix} 3 \\ 1 \end{bmatrix}\right\} \leftrightarrow 1$$

$$X(t) = \begin{bmatrix} 3e^t & e^{-t} \\ e^t & e^{-t} \end{bmatrix}$$

Hence $\begin{bmatrix} 3e^t & e^{-t} \\ e^t & e^{-t} \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{2e^t} & -\frac{1}{2e^t} \\ -\frac{1}{2e^{-t}} & \frac{3}{2e^{-t}} \end{bmatrix} = X^{-1}$

Formula (3) yields

$$x(t) = \begin{bmatrix} 3e^t & e^{-t} \\ e^t & e^{-t} \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} + \begin{bmatrix} 3e^t & e^{-t} \\ e^t & e^{-t} \end{bmatrix}\int \begin{bmatrix} \frac{1}{2e^t} & -\frac{1}{2e^t} \\ -\frac{1}{2e^{-t}} & \frac{3}{2e^{-t}} \end{bmatrix}\begin{bmatrix} e^{2t} \\ 1 \end{bmatrix} dt$$

$$= \begin{bmatrix} 3e^t & e^{-t} \\ e^t & e^{-t} \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} + \begin{bmatrix} 3e^t & e^{-t} \\ e^t & e^{-t} \end{bmatrix}\begin{bmatrix} \frac{1}{2}e^t + \frac{1}{2e^t} \\ -\frac{1}{6}e^{3t} + \frac{3}{2e^{-t}} \end{bmatrix}$$

$$= \begin{bmatrix} 3e^t & e^{-t} \\ e^t & e^{-t} \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} + \begin{bmatrix} 3e^t\left(\frac{1}{2}e^t + \frac{1}{2e^t}\right) + e^{-t}\left(-\frac{1}{6}e^{3t} + \frac{3}{2e^{-t}}\right) \\ e^t\left(\frac{1}{2}e^t + \frac{1}{2e^t}\right) + e^{-t}\left(-\frac{1}{6}e^{3t} + \frac{3}{2e^{-t}}\right) \end{bmatrix}$$

$$x(t) = \begin{bmatrix} 3e^t & e^{-t} \\ e^t & e^{-t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} + \begin{bmatrix} 3e^t\left(\frac{1}{2}e^t + \frac{1}{2e^t}\right) + e^{-t}\left(-\frac{1}{6}e^{3t} + \frac{3}{2e^{-t}}\right) \\ e^t\left(\frac{1}{2}e^t + \frac{1}{2e^t}\right) + e^{-t}\left(-\frac{1}{6}e^{3t} + \frac{3}{2e^{-t}}\right) \end{bmatrix}$$

$$x(0) = \begin{bmatrix} 3c_1 + c_2 + \frac{13}{3} \\ c_1 + c_2 + \frac{7}{3} \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

Thus

$$3c_1 + c_2 + \frac{13}{3} = -1$$
$$c_1 + c_2 + \frac{7}{3} = 0$$

, Solution is: $\left\{c_2 = -\frac{5}{6}, c_1 = -\frac{3}{2}\right\}$. Finally

$$x(t) = \begin{bmatrix} 3e^t & e^{-t} \\ e^t & e^{-t} \end{bmatrix} \begin{bmatrix} -\frac{3}{2} \\ -\frac{5}{6} \end{bmatrix} + \begin{bmatrix} 3e^t\left(\frac{1}{2}e^t + \frac{1}{2e^t}\right) + e^{-t}\left(-\frac{1}{6}e^{3t} + \frac{3}{2e^{-t}}\right) \\ e^t\left(\frac{1}{2}e^t + \frac{1}{2e^t}\right) + e^{-t}\left(-\frac{1}{6}e^{3t} + \frac{3}{2e^{-t}}\right) \end{bmatrix} =$$
$$\begin{bmatrix} -\frac{1}{6}e^{2t}\left(27e^{-t} + 5e^{-3t} - 8 - 18e^{-2t}\right) \\ -\frac{1}{6}e^{2t}\left(9e^{-t} + 5e^{-3t} - 2 - 12e^{-2t}\right) \end{bmatrix}$$

The Matrix Exponential Function (Not covered in 12F)

Recall that the general solution of the *scalar equation* $x'(t) = ax(t)$ where a is a constant is $x(t) = ce^{at}$. We will now see that the general solution of the normal system

$$x'(t) = Ax(t) \tag{1}$$

where A is a constant $n \times n$ matrix is $x(t) = e^{At}c$. We must, of course, define e^{At}.

Definition: Let A be a constant $n \times n$ matrix. The we define

$$e^{At} = I + At + A^2\frac{t^2}{2!} + \cdots + A^n\frac{t^n}{n!} + \cdots = \sum_{n=0}^{\infty} A^n\frac{t^n}{n!} \tag{2}$$

This is an $n \times n$ matrix.

Remark: If D is a diagonal matrix, then the computation of e^{Dt} is straightforward.

Example

Let $D = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}$. Then

$$D^2 = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}, \quad D^3 = \begin{bmatrix} -1 & 0 \\ 0 & 8 \end{bmatrix}, \ldots. \quad D^n = \begin{bmatrix} (-1)^n & 0 \\ 0 & 2^n \end{bmatrix}$$

Therefore

$$e^{Dt} = \sum_{n=0}^{\infty} D^n \frac{t^n}{n!} = \begin{bmatrix} \sum_{n=0}^{\infty} (-1)^n \frac{t^n}{n!} & 0 \\ 0 & \sum_{n=0}^{\infty} 2^n \frac{t^n}{n!} \end{bmatrix} = \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{2t} \end{bmatrix}$$

In general if D is an $n \times n$ diagonal matrix with $r_1, r_2, \ldots, r_n$ down its main diagonal, then e^{Dt} is the diagonal matrix with $e^{r_1 t}, e^{r_2 t}, \ldots, e^{r_n t}$ down its main diagonal.

It can be shown that the series (2) converges for all t and has many of the same properties as the scalar exponential e^{at}.

Remark: It can be shown that if a matrix A has n linearly independent eigenvectors, then $P^{-1} AP$ is a diagonal matrix, where P is formed from the n linearly independent eigenvectors of A. Thus

$$P^{-1}AP = D \tag{3}$$

where D is a diagonal matrix. In fact, D has the eigenvalues of A along its diagonal.

Let $A = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 4 & -4 & 5 \end{bmatrix}$. Then $\left\{\begin{bmatrix} -1 \\ 1 \\ 4 \end{bmatrix}\right\} \leftrightarrow 3, \left\{\begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}\right\} \leftrightarrow 1, \left\{\begin{bmatrix} -2 \\ 1 \\ 4 \end{bmatrix}\right\} \leftrightarrow 2$

Thus $P = \begin{bmatrix} -1 & -1 & -2 \\ 1 & 1 & 1 \\ 4 & 2 & 4 \end{bmatrix}$, inverse: $\begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 2 & -\frac{1}{2} \\ -1 & -1 & 0 \end{bmatrix} = P^{-1}$

Hence $P^{-1}AP = \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 2 & -\frac{1}{2} \\ -1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 4 & -4 & 5 \end{bmatrix} \begin{bmatrix} -1 & -1 & -2 \\ 1 & 1 & 1 \\ 4 & 2 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

Now (3) implies that when A has n linearly independent eigenvalues we have

$$A = PDP^{-1}$$

so that

$$
\begin{aligned}
e^{At} &= e^{PDP^{-1}t} = I + PDP^{-1}t + \frac{1}{2}\left(PDP^{-1}t\right)\left(PDP^{-1}t\right) + \cdots \\
&= I + PDP^{-1}t + \frac{1}{2}\left(PDP^{-1}\right)\left(PDP^{-1}\right)t^2 + \cdots \\
&= I + PDP^{-1}t + \frac{1}{2}\left(PD^2P^{-1}\right)t^2 + \cdots \\
&= P\left(I + Dt + \frac{1}{2}(Dt)^2 + \cdots\right)P^{-1} \\
&= Pe^{Dt}P^{-1}
\end{aligned}
$$

Example:

Let $A = \begin{bmatrix} 5 & 4 \\ -2 & -1 \end{bmatrix}$,, eigenvectors: $\left\{\begin{bmatrix} -2 \\ 1 \end{bmatrix}\right\} \leftrightarrow 3, \left\{\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right\} \leftrightarrow 1$

$P = \begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix}, P^{-1} = \begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix}$ and $D = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$. Show that $A = P^{-1}DP$ and use this to compute e^{At}.

$A = PDP^{-1} = \begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 5 & 4 \\ -2 & -1 \end{bmatrix}$ as required.

Thus $e^{\begin{bmatrix} 5 & 4 \\ -2 & -1 \end{bmatrix}t} = \begin{bmatrix} -e^t + 2e^{3t} & 2e^{3t} - 2e^t \\ -e^{3t} + e^t & 2e^t - e^{3t} \end{bmatrix}$ from SNB.

Also $Pe^{Dt}P^{-1} = \begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix} e^{\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}t} \begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix}$ so

$e^{At} = Pe^{Dt}P^{-1} = \begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} e^t & 0 \\ 0 & e^{3t} \end{bmatrix}\begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} -e^t + 2e^{3t} & -2e^t + 2e^{3t} \\ e^t - e^{3t} & 2e^t - e^{3t} \end{bmatrix}$

This e^{At} is a fundamental matrix for the system

$$x'(t) = \begin{bmatrix} 5 & 4 \\ -2 & -1 \end{bmatrix} x(t)$$

since if we let

$$x_h(t) = e^{\begin{bmatrix} 5 & 4 \\ -2 & -1 \end{bmatrix} t} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} -e^t + 2e^{3t} & -2e^t + 2e^{3t} \\ e^t - e^{3t} & 2e^t - e^{3t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$= \begin{bmatrix} c_2\left(2e^{3t} - 2e^t\right) - c_1\left(e^t - 2e^{3t}\right) \\ c_1\left(e^t - e^{3t}\right) - c_2\left(e^{3t} - 2e^t\right) \end{bmatrix}$$

$$= \begin{bmatrix} c_2\left(2e^{3t} - 2e^t\right) - c_1\left(e^t - 2e^{3t}\right) \\ c_1\left(e^t - e^{3t}\right) - c_2\left(e^{3t} - 2e^t\right) \end{bmatrix}$$

then

$$x_h'(t) = \begin{bmatrix} c_2\left(6e^{3t} - 2e^t\right) - c_1\left(e^t - 6e^{3t}\right) \\ c_1\left(e^t - 3e^{3t}\right) - c_2\left(3e^{3t} - 2e^t\right) \end{bmatrix}$$

and

$$Ax_h(t) = \begin{bmatrix} 5 & 4 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} c_2\left(2e^{3t} - 2e^t\right) - c_1\left(e^t - 2e^{3t}\right) \\ c_1\left(e^t - e^{3t}\right) - c_2\left(e^{3t} - 2e^t\right) \end{bmatrix}$$

$$= \begin{bmatrix} 4c_1\left(e^t - e^{3t}\right) - 5c_1\left(e^t - 2e^{3t}\right) - 4c_2\left(e^{3t} - 2e^t\right) + 5c_2\left(2e^{3t} - 2e^t\right) \\ 2c_1\left(e^t - 2e^{3t}\right) - c_1\left(e^t - e^{3t}\right) + c_2\left(e^{3t} - 2e^t\right) - 2c_2\left(2e^{3t} - 2e^t\right) \end{bmatrix}$$

$$= \begin{bmatrix} 6c_1e^{3t} - 2c_2e^t - c_1e^t + 6c_2e^{3t} \\ c_1e^t + 2c_2e^t - 3c_1e^{3t} - 3c_2e^{3t} \end{bmatrix} = \begin{bmatrix} c_2\left(6e^{3t} - 2e^t\right) - c_1\left(e^t - 6e^{3t}\right) \\ c_1\left(e^t - 3e^{3t}\right) - c_2\left(3e^{3t} - 2e^t\right) \end{bmatrix} = x_h'(t)$$

:

Theorem 5 (Properties of the Matrix Exponential Function)
Let A and B be $n \times n$ constant matrices and r, s and t be real (or complex) numbers. Then

1. **a.** $e^{A0} = e^0 = I$
b. $e^{A(t+s)} = e^{At}e^{As}$
c. $\left(e^{At}\right)^{-1} = e^{-At}$
d. $e^{(A+B)t} = e^{At}e^{Bt}$ provided that $AB = BA$
e. $e^{rIt} = e^{rt}I$
f. $\frac{d}{dt}\left(e^{At}\right) = Ae^{At}$

Remark: c. tells us that the matrix e^{At} has an inverse.

Proof of f.

$$\frac{d}{dt}\left(e^{At}\right) = \frac{d}{dt}\left(I + At + A^2\frac{t^2}{2!} + \cdots + A^n\frac{t^n}{n!} + \cdots\right)$$
$$= A + A^2t + A^3\frac{t^2}{2!} + \cdots + A^n\frac{t^{n-1}}{(n-1)!} + \cdots = A\left[I + At + A^2\frac{t^2}{2!} + \cdots + A^n\frac{t^n}{n!} + \cdots\right]$$
$$= Ae^{At}$$

Theorem 6 (**e^{At} is a Fundamental Matrix**)
If A is an $n \times n$ constant matrix, then the columns of the matrix e^{At} form a fundamental solution set for the system $x'(t) = Ax(t)$. Therefore, e^{At} is a fundamental matrix for the system, and a general solution is $x(t) = e^{At}c$.

Lemma (**Relationship Between Fundamental Matrices**)
Let $X(t)$ and $Y(t)$ be two fundamental matrices for the same system $x' = Ax$. Then there exists a constant column matrix C such that $Y(t) = X(t)C$.

Remark: Let $Y(t) = e^{At} = X(t)C$ and set $t = 0$. Then
$$I = X(0)C \quad \Rightarrow \quad C = X(0)^{-1}$$
and
$$e^{At} = X(t)X(0)^{-1}$$
If the *nxn* matrix A has n linearly independent eigenvectors u_i, then $[e^{r_1t}u_{1,}e^{r_2t}u_2, \ldots, e^{r_nt}]$ is a fundamental matrix for $x' = Ax$ and

$$e^{At} = [e^{r_1t}u_{1,}e^{r_2t}u_2, \ldots, e^{r_nt}][u_1, u_2, \ldots, u_n]^{-1}$$

Calculating e^{At} for Nilpotent Matrices

Definition: An *nxn* matrix A matrix is nilpotent if for some positive integer k
$$A^k = 0.$$

Since
$$e^{At} = I + At + A^2\frac{t^2}{2!} + \cdots + A^n\frac{t^n}{n!} + \cdots = \sum_{n=0}^{\infty} A^n\frac{t^n}{n!}$$
we see that if A is nilpotent, then the infinite series has only a finite number of terms since $A^k = A^{k+1} = \cdots = 0$ and in this case
$$e^{At} = I + At + A^2\frac{t^2}{2!} + \cdots + A^{k-1}\frac{t^{k-1}}{(k-1)!}$$
This may be taken further. The Cayley-Hamilton Theorem says that a matrix satisfies its own characteristic equation, that is, $p(A) = 0$. Therefore, if the characteristic polynomial for A has the form $p(r) = (-1)^n(r - r_1)^n$, that is A has only one multiple eigenvalue r_1, then $p(A) = (-1)^n(A - r_1I)^n = 0$. Hence $A - r_1I$ is nilpotent and
$$e^{At} = e^{(r_1I + A - r_1I)t} = e^{r_1It}e^{(A - r_1I)t} = e^{r_1t}\left[I + (A - r_1I)t + \cdots + (A - r_1I)^{n-1}\frac{t^{n-1}}{(n-1)!}\right]$$
Example Find the fundamental matrix e^{At} for the system

$$x'(t) = Ax(t) \text{ where } A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ -2 & -2 & -1 \end{bmatrix}$$

Solution: The characteristic polynomial for A is

$$p(r) = \det\begin{bmatrix} 2-r & 1 & 1 \\ 1 & 2-r & 1 \\ -2 & -2 & -1-r \end{bmatrix} = -r^3 + 3r^2 - 3r + 1 = -(r-1)^3$$

Hence $r = 1$ is an eigenvalue of A with multiplicity 3. By the Cayley-Hamilton Theorem $(A-I)^3 = 0$ and

$$e^{At} = e^t e^{(A-I)t} = e^t\left[I + (A-I)t + (A-I)^2 \frac{t^2}{2!} \right]$$

$$A - I = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ -2 & -2 & -2 \end{bmatrix} \text{ and } (A-I)^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus

$$e^{At} = e^t\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + te^t\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ -2 & -2 & -2 \end{bmatrix} = \begin{bmatrix} e^t + te^t & te^t & te^t \\ te^t & e^t + te^t & te^t \\ -2te^t & -2te^t & e^t - 2e^t \end{bmatrix}$$